AN M/G/1 QUEUEING SYSTEM WITH
COMPULSORY SERVER VACATIONS

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ABSTRACT

The paper deals with a single server queue with compulsory server vacations. There are single Poisson arrivals with mean arrival rate $\lambda$ and the service is performed in batches of fixed size $M(\geq 1)$ or min $(n, M)$ and the service times of successive batches follow a general distribution with the density function $D(x)$. The Laplace transforms of the probability generating functions of different states of the system have been obtained, the corresponding steady state results have been derived and in a particular case the mean queue length has been obtained explicitly.

Key words: Poisson arrivals, batch service, arbitrary (general) service times, server vacations, probability generating function, Laplace transform, idle state, steady state, mean queue length.

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1. INTRODUCTION

Queueing models with vacations have been investigated by many authors including Keilson and Servi [4], Cramer [1], Scholl and Kleinrock [7], Shanthikumar [8], Doshi [2] and Madan [5, 6]. The present paper studies a single server queue with Poisson input, bulk output with arbitrary (general) service times and compulsory server vacations. Such situations can be visualized in queueing systems where after each service

the system may demand a compulsory pause for overhauling or for warming up or for loading, etc. The following assumptions briefly describe the mathematical model:

I. Customers arrive at the system one by one in a Poisson stream with mean arrival rate \( \lambda \).

II. The service of units is rendered in batches of fixed size \( M \geq 1 \) or \( \min(n, M) \), where \( n \) is the number of customers in the queue and the service times of successive batches follow a general distribution with the density function \( D(x) \).

III. Server vacation starts as soon as the service of a batch is completed. The duration of the vacation period is assumed to be exponential with mean vacation time \( 1/b \). In other words \( b \Delta t \) is the first order probability that the server will be back during the time interval \( (t, t + \Delta t) \).

2. EQUATIONS GOVERNING THE SYSTEM

We define \( W_n(x, t) \) the probability that at time \( t \) the server is working and there are \( n \geq 0 \) customers in the queue, excluding a batch in service with lapsed service time lying between \( x \) and \( x + dx \). \( V_n(t) \) as the probability that at time \( t \) there are \( n \geq 0 \) customers in the queue and the server is on vacation and \( Q(t) \) is the probability that the server is idle but available (present in the system).

The system is governed by the following set of differential-difference equations:

\[
\frac{\partial}{\partial x} W_n(x, t) + \frac{\partial}{\partial t} W_n(x, t) + (\lambda + \mu(x))W_n(x, t) = \lambda W_{n-1}(x, t) \quad (n > 0) \quad (1)
\]

\[
\frac{\partial}{\partial x} W_0(x, t) + \frac{\partial}{\partial t} W_0(x, t) + (\lambda + \lambda + \mu(x))W_0(x, t) = 0 \quad (2)
\]

\[
\frac{d}{dt} V_n(t) + (\lambda + b)V_n(t) = \lambda V_{n-1}(t) + \int_0^\infty W_n(x, t)\mu(x)dx \quad (n > 0) \quad (3)
\]

\[
\frac{d}{dt} V_0(t) + (\lambda + b)V_0(t) = \int_0^\infty W_0(x, t)\mu(x)dx \quad (4)
\]
\[
\frac{d}{dt} Q(t) + \lambda Q(t) = b V_0(t)
\]  

(5)

where \(\mu(x)\Delta x\) is the first order probability that the service of a batch will be completed in time \(x\) and \(x + \Delta x\) conditioned that the same was not completed till time \(x\) and is related to \(D(x)\) by the relation

\[
D(x) = \mu(x) \exp \left( - \int_0^x \mu(t) dt \right)
\]  

(6)

The above equations are to be solved subject to the boundary conditions

\[
W_n(0, t) = b V_{n+M}(t) \quad (n > 0)
\]  

(7)

\[
W_0(0, t) = \lambda Q(t) + b \sum_{r=1}^{M} V_r(t)
\]  

(8)

We assume that initially the server is available but idle because of no customers so that the initial condition is

\[
Q(0) = 1
\]  

(9)

3. THE TIME DEPENDENT SOLUTION

We define the following probability generating functions:

\[
W(x, z, t) = \sum_{n=0}^{\infty} z^n W_n(x, t) \quad , \quad W(z, t) = \sum_{n=0}^{\infty} z^n W_n(t)
\]  

(10)

\[
V(z, t) = \sum_{n=1}^{\infty} z^n V_n(t)
\]

where \(W_n(t)\) is the probability that at time \(t\) there are \(n\) units in the queue and a batch of units is being served irrespective of the value of \(x\) so that

\[
W_n(t) = \int_0^{\infty} W_n(x, t) dx
\]  

(11)
and, therefore,

\[ W(z, t) = \int_{0}^{\infty} W(x, z, t) dx \]  

(12)

Define the Laplace transform of a function \( f(t) \) as follows:

\[ \mathcal{L}[f(t)] = \int_{0}^{\infty} e^{-st} f(t) \, dt, \quad \text{Re}(s) > 0 \]  

(13)

We note that the Laplace transform of \( \frac{d}{dt} f(t) \) is given by

\[ \mathcal{L}\left[ \frac{d}{dt} f(t) \right] = s\mathcal{L}(f) - f(0) \]  

(14)

We take Laplace transform of equations (1)-(5) and (7)-(8) and use (9), (13) and (14) and have

\[ \frac{\partial}{\partial x} \mathcal{W}_n(x, s) + (s + \lambda + \mu(x))\mathcal{W}_n(x, s) = \lambda \mathcal{W}_{n-1}(x, s) \quad (n > 0) \]  

(15)

\[ \frac{\partial}{\partial x} \mathcal{W}_0(x, s) + (s + \lambda + \mu(x))\mathcal{W}_0(x, s) = 0 \]  

(16)

\[ (s + \lambda + b)\mathcal{V}_n(s) = \lambda \mathcal{V}_{n-1}(s) + \int_{0}^{\infty} \mathcal{W}_n(x, s) \mu(x) dx \quad (n > 0) \]  

(17)

\[ (s + \lambda + b)\mathcal{V}_0(s) = \int_{0}^{\infty} \mathcal{W}_0(x, s) \mu(x) dx \]  

(18)

\[ (s + \lambda)\mathcal{Q}(s) = 1 + b\mathcal{V}_0(s) \]  

(19)

\[ \mathcal{W}_n(0, s) = b\mathcal{V}_{n+M}(s) \quad (n > 0) \]  

(20)

\[ \mathcal{W}_0(0, s) = \lambda\mathcal{Q}(s) + b \sum_{r=1}^{M} \mathcal{V}_r(s) \]  

(21)

We perform

\[ \sum_{n=1}^{\infty} z^n(15) + (16) ; \quad \sum_{n=1}^{\infty} z^n(17) + (18) \quad \text{and} \quad \sum_{n=1}^{\infty} (20) + (21) \]
make use of (10) and simplify. Thus we have

\[ \frac{\partial}{\partial x} \bar{W}(x, z, s) + [s + \lambda - \lambda z + \mu(x)] \bar{W}(x, z, s) = 0 \]  

(22)

\[ [s + \lambda - \lambda z + b] \bar{V}(z, s) = \int_0^\infty \bar{W}(x, z, s) \mu(x) dx \]  

(23)

\[ z^M \bar{W}(0, z, s) = b \bar{V}(z, s) + \lambda z^M \bar{Q}(s) + b \sum_{r=1}^{M-1} (z^M - z^r) \bar{V}_r(s) - b \bar{V}_0(s) \]  

(24)

Equation (24), on using (19), gives

\[ \bar{V}(0, z, s) = b z^{-M} \bar{P}(z, s) + (\lambda - (s + \lambda) z^{-M}) \bar{Q}(s) + z^{-M} + \sum_{r=1}^{M-1} (1 - z^{-M+r}) \bar{V}_r(s) \]  

(25)

We integrate equation (22) between the limits 0 and x and have

\[ \bar{W}(x, z, s) = \bar{W}(0, z, s) \exp \left[-[s + \lambda - \lambda z] x - \int_0^x \mu(t) dt \right] \]  

(26)

where \( \bar{W}(0, z, s) \) is given by (25).

Integrate equation (26) w.r.t. x, we have

\[ \bar{W}(s, z) = \bar{W}(0, z, s) \left[ \frac{1 - \bar{D}(s + \lambda - \lambda z)}{s + \lambda - \lambda z} \right] \]  

(27)

where \( \bar{D}(s + \lambda - \lambda z) = \int_0^\infty e^{-(s + \lambda - \lambda z)x} D(x) dx \).

Now, using the value of \( \bar{W}(x, z, s) \) from (26) in (23) and using (25) we have

\[ [s + \lambda - \lambda z + b] \bar{V}(z, s) = \]  

\[ = \left[ b z^{-M} \bar{P}(z, s) + (\lambda - (s + \lambda) z^{-M}) \bar{Q}(s) + z^{-M} + \sum_{r=1}^{M-1} (1 - z^{-M+r}) \bar{V}_r(s) \right] = \]  

\[ \times \left[ \int_0^\infty \exp \left[-(s + \lambda - \lambda z) x - \int_0^x \mu(t) dt \right] \mu(x) dx \right] \]  

(28)
Clearly, the integral on the right side of equation (28) equals \( \bar{D}[s + \lambda - \lambda z] \) by virtue of equations (6) and (13). Therefore, (28) can be written as

\[
[s + \lambda - \lambda z + b] \bar{V}(z, s) = \left[ b z^{-M} \bar{V}(z, s) + (\lambda - (s + \lambda)z^{-M}) \bar{Q}(s) + z^{-M} + \sum_{r=1}^{M-1} (1-z^{-M+r}) \bar{V}(s) \right] \times \bar{D}[s + \lambda - \lambda z] \tag{29}
\]

which, on simplifying, yields

\[
\bar{V}(s, z) = \frac{[\lambda(z^M - 1) - s] \bar{Q}(s) + 1 + \sum_{r=1}^{M-1} (z^M - z^r) \bar{V}(s)}{[s + \lambda - \lambda z + b]z^M - b \bar{D}[s + \lambda - \lambda z]} \tag{30}
\]

We note that there are \( M \) unknowns, \( \bar{Q}(s) \) and \( \bar{V}_r(s), r = 1, 2, \ldots, M - 1 \), appearing in equation (30). In order to determine these zeroes we apply Rouche's theorem as follows:

Let \( f(z) = z^M \) and \( g(z) = \frac{b \bar{D}[s + \lambda - \lambda z]}{s + \lambda - \lambda z + b} \). Both \( F(z) \) and \( g(z) \) are differentiable inside and continuous on the contour \( |z| = 1 \). Also, on \( |z| = 1, |f(z)| = 1 \) and

\[
|g(z)| = \left| \frac{b \bar{D}[s + \lambda - \lambda z]}{s + \lambda - \lambda z + b} \right|
\]

\[
= \left| \frac{b}{s + \lambda - \lambda z + b} \right| \left| \bar{D}[s + \lambda - \lambda z] \right|
\]

\[
= \left| \frac{b}{s + \lambda - \lambda z + b} \right| \left| \int_0^\infty e^{-(s + \lambda - \lambda z)x} D(x) dx \right|
\]

\[
= \left| \frac{b}{s + \lambda - \lambda z + b} \right| \int_0^\infty \frac{|D(x)|}{e^{(s + \lambda - \lambda z)x}} dx
\]

\[
= \left| \frac{b}{s + \lambda - \lambda z + b} \right| \int_0^\infty |D(x)| dx
\]

\[
= \left| \frac{b}{s + \lambda - \lambda z + b} \right| \cdot 1
\]

\[
< 1 = |f(z)|
\]
Thus on $|z| = 1$, $|f(z)| > |g(z)|$. Hence by Rouche’s Theorem $f(z) - g(z)$ has the same number of zeroes as that of $f(z)$ inside the unit circle $|z| = 1$. Clearly $f(z)$ has $M$ zeroes inside the unit circle and, therefore, $f(z) - g(z)$ i.e. the denominator in the right hand side of equation (30) has $M$ zeroes inside the unit circle $|z| = 1$.

Since $ar{W}(s, z)$ is regular inside the contour $|z| = 1$, the numerator in the right hand side of equation (30) must vanish for these zeroes of the denominator, giving rise to a set of $M$ linear equations which are sufficient to determine the $M$ unknowns. Thus $\bar{W}(s, z)$ and, for that matter, $\bar{W}(s, z)$ can be completely determined.

We would continue the analysis for the particular case when the service times follow an exponential distribution with parameter $\mu$, then

$$D(x) = \mu e^{-\mu x}, \quad x > 0$$

$$\bar{D}(s + \lambda - \lambda z) = \int_0^\infty e^{-(s + \lambda - \lambda z) t} \mu e^{-\mu t} dt = \frac{\mu}{s + \lambda - \lambda z + \mu}$$

With this substitution for $\bar{D}(s + \lambda - \lambda z)$, equation (30) yields

$$\bar{P}(s, z) = \frac{[(s + \lambda - \lambda z - 1) - s]Q(s) + 1 + \sum_{r=1}^{M-1} (z^M - z^r) P_r(s)}{[s + \lambda - \lambda z + \mu][s + \lambda - \lambda z + b]z^M - b\mu}$$ (31)

and from (27) we have

$$\bar{W}(s, z) = \frac{\bar{W}(0, z, s)}{s + \lambda - \lambda z + \mu}$$ (32)

4. THE STEADY STATE RESULTS

To define the steady state probabilities and corresponding generating functions, we drop the argument $t$, and for that matter the argument $s$ wherever it appears in the time-dependent analysis up to this point. Then the corresponding steady state results can be obtained by using the well known Tauberian property

$$\lim_{s \to 0} s \bar{f}(s) = \lim_{t \to \infty} f(t)$$ (33)

if the limit on the right exists.
Thus multiplying both sides of (31) by $s$ and using (33), we have

$$V(z) = \frac{(\lambda(z^M - 1)Q + \sum_{r=1}^{M-1} (z^M - z^r)V_r)\mu}{[\lambda - \lambda z + \mu][\lambda - \lambda z + b]z^M - b\mu}$$  \hspace{1cm} (34)$$

and from (25), we have

$$W(0, z) = b z^{-M} V(z) + \lambda(1 - z^{-M})Q + \sum_{r=1}^{M-1} (1 - z^{-M+r})V_r$$  \hspace{1cm} (35)$$

so that from (32), we have

$$W(z) = \frac{b z^{-M} V(z) + \lambda(1 - z^{-M})Q + \sum_{r=1}^{M-1} (1 - z^{-M+r})V_r}{\lambda - \lambda z + \mu}$$  \hspace{1cm} (36)$$

The $M$ unknowns, $Q$ and $V_r$, $r = 1, 2, ..., M - 1$ can be determined as before.

We note that for $z = 1$, the right hand side of (34) is indeterminate of $\frac{0}{0}$ form. Using L'Hôpital's rule we have

$$\lim_{z \to 1} V(z) = \frac{[\lambda M z^{M-1}Q + \sum_{r=1}^{M-1} (Mz^{M-1} - rz^{r-1})V_r]\mu}{\mu b M z^{M-1} - \lambda b - \lambda \mu}$$  \hspace{1cm} (37)$$

so that

$$V(1) = \frac{[\lambda MQ + \sum_{r=1}^{M-1} (M - r)V_r]\mu}{\mu b M - \lambda b - \lambda \mu}$$  \hspace{1cm} (38)$$

This is the probability that the server is on vacation irrespective of the number of customers in the queue. Using (38) in (35) we have

$$W(1) = \frac{[\lambda MQ + \sum_{r=1}^{M-1} (M - r)V_r]\mu}{\mu b M - \lambda b - \lambda \mu}$$  \hspace{1cm} (39)$$
This is the probability that the server is working irrespective of the number of customers in the queue.

5. A PARTICULAR CASE

If \( M = 1 \), i.e. if the service is performed one by one, then from (38), (39) we have

\[
V(1) = \frac{\lambda \mu Q}{\mu b - \lambda (b + \mu)} \tag{40}
\]

\[
W(1) = \frac{\lambda \mu Q}{\mu b - \lambda (b + \mu)} \tag{41}
\]

Since \( V(1) + W(1) + Q = 1 \), we have from (40) and (41)

\[
Q = 1 - \lambda \left( \frac{1}{\mu} + \frac{1}{b} \right) \tag{42}
\]

the probability that the system is available but idle, which also seems intuitively true.

We derive the steady state results corresponding to \( M = 1 \) from equations (34) and (36) which, on simplifying and using (42), yield

\[
V(z) = \frac{\lambda \mu (z - 1) \left[ 1 - \lambda \left( \frac{1}{\mu} + \frac{1}{b} \right) \right]}{(\lambda - \lambda z + \mu)(\lambda - \lambda z + b)z - b\mu} \tag{43}
\]

\[
W(z) = \frac{\lambda (z - 1)(\lambda - \lambda z + b) \left[ 1 - \lambda \left( \frac{1}{\mu} + \frac{1}{b} \right) \right]}{(\lambda - \lambda z + \mu)(\lambda - \lambda z + b)z - b\mu} \tag{44}
\]

6. THE MEAN QUEUE LENGTH

Now, the common denominator of the right hand side of equations (43) and (44) can be factored as \((z - 1)\left[ \lambda^2 - \lambda(\lambda + b + \mu)z + \mu b \right]\), so that the factor \((z - 1)\) can be cancelled out with the numerator and, therefore, equations (43) and (44) can be re-written as
\begin{align*}
V(z) &= \frac{\lambda \mu \left[1 - \lambda \left(\frac{1}{\mu} + \frac{1}{b}\right)\right]}{\lambda^2 - \lambda(\lambda + b + \mu)z + \mu b} \\
W(z) &= \frac{\lambda(\lambda - \lambda z + b)\left[1 - \lambda \left(\frac{1}{\mu} + \frac{1}{b}\right)\right]}{\lambda^2 - \lambda(\lambda + b + \mu)z + \mu b}
\end{align*}
(45)

Let $L_v$ denote the mean queue length when the server is on vacation and $L_w$ the mean queue length (excluding those in service) when the server is busy, then

\begin{align*}
L_q &= \sum_{n=0}^{\infty} nV_n = \frac{d}{dz} V(z) \bigg|_{z=1} \\
L_w &= \sum_{n=0}^{\infty} nW_n = \frac{d}{dz} W(z) \bigg|_{z=1}
\end{align*}
(47)

Evaluating the derivatives at $z = 1$ and simplifying we have, finally

\begin{align*}
L_v &= \frac{\lambda^2 \mu(b + \mu - \lambda) \left[1 - \lambda \left(\frac{1}{\mu} + \frac{1}{b}\right)\right]}{[\mu b - \lambda(b + \mu)]^2} \\
L_w &= \frac{\lambda^2(\lambda \mu + b^2) \left[1 - \lambda \left(\frac{1}{\mu} + \frac{1}{b}\right)\right]}{[\mu b - \lambda(b + \mu)]^2}
\end{align*}
(49)

Let $L$ denote the mean queue length irrespective of whether the server is working or on vacation, then $L = L_v + L_w$ and equations (49) and (50), on adding and simplifying yield

\begin{align*}
L &= \frac{\lambda^2(b^2 + \mu^2 + \mu b) \left[1 - \lambda \left(\frac{1}{\mu} + \frac{1}{b}\right)\right]}{[\mu b - \lambda(b + \mu)]^2}, \quad \lambda < \frac{b \mu}{b + \mu}
\end{align*}
(51)

We note that, if there are no server vacations then $\frac{1}{b} = 0$. Dividing the numerator and the denominator of equation (51) by $b^2$ and letting
\[ \frac{1}{b} \rightarrow 0 \text{ and simplifying we finally have} \]

\[ L = \frac{\lambda^2}{\mu^2} \left( 1 - \frac{\lambda}{\mu} \right) \quad (52) \]

which is a well-known result.

REFERENCES


