CONTROL OF LINEAR SYSTEMS WITH RATIONAL EXPECTATIONS. THE CASE OF INCOMPLETE INFORMATION

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SUMMARY

The problem of optimal control of linear economic systems with rational expectations and quadratic objective function is solved for the case of incomplete information. The case of complete information has been previously studied. In both problems the hypothesis of causality is not satisfied and, therefore, the standard techniques of control theory cannot be directly applied, though the method used is based on these techniques.

Key words: Incomplete information, Rational expectations, Stochastic control.
AMS Classification: 93A99.

1. INTRODUCTION

In most non-trivial economic decisions the time variable intervenes. Many economic agents' decisions at a particular moment in time depend on their view of the future, that is to say, on their expectations. The same applies with the behaviour of the economic aggregates making up macroeconomic models. In the economic literature there exist different ways of modelling expectations, the most important being: static expectations, adaptive and rational ones, with the latter having acquired great importance in recent years, and now being a necessary reference in any treatise on Dynamic Economics (Pesaran (1987), Aoki (1989), Holly and Hughes Hallet (1989)).
The rational expectations hypothesis, in Muth’s strong version, assumes that the expectations that the economic agents has at moment $t$, on the value that a variable will take in the future (which is subjective and non-observable), is the mathematical expectation of the variable conditioned by the information which is held in $t$, implied by the model. The hypothesis assumes, therefore, that individuals act as if they knew the model and formed their expectations in accordance with it.

The control theory techniques are not applicable, in general, to systems with rational expectations of the state variables, since the causality hypothesis is not met. (Aoki-Canzoneri (1979), Chow (1980), Driffill (1981), Buiter (1983)).

In Cerdá (1990) a method is presented, based on Dynamic Programming, which solves the control problem of a linear systems with rational expectations and a quadratic objective function. In this article we are going to deal with the problem for the case of incomplete information: it is assumed that the state variables (endogenous variables) are not observable, but there exist observation variables related to them by the observation equation.

In section 2 the problem is set out. In section 3 some preliminary results are obtained. In section 4 the theorem for solving the problem is presented. In section 5 the article’s conclusions are presented.

2. STATEMENT OF THE PROBLEM

We consider the problem with the following objective function, state equation and observation equation:

$$
\min E_0 W = E_0 \sum_{t=1}^{T} (y_t - a_t)K_t(y_t - a_t), \quad K_t \text{ being a positive semidefinite, symmetric matrix.}
$$

$$
y_t = B_{it} y_{i,t-1}^* + B_{ir} y_{i,r+1,t-1} + A_{it} y_{t-1} + C_i x_t + b_t + u_t \\
(\text{for } t = 1, \ldots, T)
$$

$$
z_t = M_i y_t + w_t \\
(\text{for } t = 0, \ldots, T)
$$

where:

$y_t$ is a non-observable vector of endogenous variables.
$x_t$ is a vector of control variables.

$b_t$ is a vector which records the combined effects of exogenous variables not subject to control.

$z_t$ is a vector of observable variables.

We assume that $y_{0}, u_{1}, ..., u_{T}, w_{0}, ..., w_{T}$ are random mutually uncorrelated vectors, so that:

$$Eu_t = 0 \quad ; \quad Eu_t u_t' = U_t$$

$$Ew_t = 0 \quad ; \quad Ew_t w_t' = W_t$$

$$Ey_0 = m \quad ; \quad E(y_0 - m)(y_0 - m)' = S$$

Moreover, we assume that $W_t^* = E(y_t | I_k)$ is the (rational) expectation that is held at the end of the period $k$ on the value that the vector $y$ will take in period $t$. In this case $I_k = \{z_k, ..., z_0; x_k, ..., x_1; \ b_0, ..., b_1\}$ but does not contain $y_k, ..., y_0$, since they are unknown (non-observable).

$$E_0 W = E(W | I_0)$$

We assume that the exogenous variables $\{b_t\}$ are stochastic, in the form:

$$b_t = \sum_{i=1}^{p} R_i b_{t-i} + \xi_t$$

where $\{\xi_t\}$ is a serially uncorrelated stochastic process, of zero mean, which is independent of the disturbances that enter the system explaining $y_t$, of the observation noises and of the random variable $y_0$ ($p$ can be considered finite or infinite).

**Note:** In its most general form the system (1) would be,

$$y_t = B_0 y_{\infty t-1} + \sum_{i=1}^{p} B_{i} y_{i+t-1} + A_1 y_{t-1} + C_t x_t + b_t + u_t \quad (\text{for } t = 1, ..., T)$$

In this article the case $p = 1$ is developed, since if $p \neq 1$ the treatment is analogous.
3. PRELIMINARY RESULTS

3.1. Expression of the system in another equivalent form

Proposition 1

Let us consider system (1). Let us assume that, for each \( t \), it is verified that the matrix \( (I - B_t) \) is non singular. Thus, this system can be explained in the following way

\[
y_t = B_{1t}y_{t-1}^* + A_t y_{t-1} + (\Delta_t - A_t)E(y_{t-1}|I_{t-1}) + C_t x_{t-1}^* + \delta_{t-1}^* + v_t
\]

where:

\[
B_{1t} = (I - B_t)^{-1}B_{1t} \\
\Delta_t = (I - B_t)^{-1}A_t \\
C_t = (I - B_t)^{-1}C_t \\
\delta_{t-1}^* = (I - B_t)^{-1}\delta_{t-1}^* \\
v_t = C_t(x_t - x_{t-1}^*) + (b_t - b_{t-1}^*) + u_t 
\]

Proof:

We consider system (1)

\[
y_t = B_{1t}y_{t-1}^* + B_{1t}y_{t-1}^* + A_t y_{t-1} + C_t x_t + b_t + u_t \quad \text{(for } t = 1, 2, ..., T) \]

where, as we have pointed out in section 2, expectations are rational, so that \( y_{t-1}^* = E[y_{t-1}|I_{t-1}] \); \( y_{t-1}^* = E[y_{t-1}|I_{t-1}] \).

Taking the expectations conditioned to \( I_{t-1} \) on the two sides of system (1) and bearing in mind that the expectations are rational, we obtain:

\[
y_{t-1}^* = B_{1t}y_{t-1}^* + B_{1t}y_{t-1}^* + A_t E(y_{t-1}|I_{t-1}) + C_t x_{t-1}^* + \delta_{t-1}^* \Rightarrow \Rightarrow y_{t-1}^* = (I - B_t)^{-1}[B_{1t}y_{t-1}^* + A_t E(y_{t-1}|I_{t-1}) + C_t x_{t-1}^* + \delta_{t-1}^*] 
\]

therefore:

\[
y_t = y_{t-1}^* + A_t y_{t-1} - A_t E(y_{t-1}|I_{t-1}) + v_t = B_{1t} y_{t-1}^* + A_t y_{t-1} + (\Delta_t - A_t)E(y_{t-1}|I_{t-1}) + C_t x_{t-1}^* + \delta_{t-1}^* + v_t 
\]
Corollary

In the case of complete information, we have

\[ I_k = \{ y_k, \ldots, y_0; x_k, \ldots, x_1; b_k, \ldots, b_1 \} \]

so that

\[ y_t = B_{1t} y_{t+1; t-1} + A_t y_{t-1} + C_t x_{t-1} + F_{t-1} + v_t \]

a result that coincides with the one we had in that case (Cerdà, 1990).

Note: We assume, as in the case of complete information, that the vectors \( v_t \) are uncorrelated in time and have zero mean. Moreover, \( y_0, v_1, \ldots, v_T, w_0, \ldots, w_T \) are mutually uncorrelated. Hence \( E v_t = 0 \); we call \( E v_t v_t' = V_t \).

### 3.2. Previous problem

Before setting out and proving the theorem that solves the problem we are dealing with, we are going to raise another problem of control with incomplete information, for a particular formulation of the system, without expectations, the solution of which we will use in an auxiliary way in proving the theorem which most concerns us.

The «previous problem» is the following one:

\[
\min E_0 W = E_0 \sum_{t=1}^{T} (y_t - a_t) K_t (y_t - a_t), \quad K_t \text{ being a positive semidefinite, symmetric matrix.}
\]

\[ y_t = D_t y_{t-1} + (A_t - D_t) E(y_{t-1}) + C_t x_t + b_t + u_t \quad \text{for} \quad t = 1, \ldots, T \]

\[ z_t = M_t y_t + w_t \quad \text{for} \quad t = 0, \ldots, T \]

We assume that \( y_0, u_1, \ldots, u_T, w_0, \ldots, w_T \) are uncorrelated. Moreover, \( E u_t = 0; E v_t = 0 \ \forall t \)

\[ I_k = \{ z_k, \ldots, z_0; x_k, \ldots, x_1; b_k, \ldots, b_1 \} \]

The exogenous variables \( \{ b_t \} \) are of the form:

\[ b_t = \sum_{i=1}^{q} R_i b_{t-i} + \eta_t \]
where \( \{ \eta_t \} \) is a stochastic process, of zero mean, serially uncorrelated, which is independent of the disturbances that enter the system that explains \( y_t \) of the observation noises and of the random variable \( y_0 \).

### 3.3. Solution of the previous problem

**Theorem 1**

The solution to the problem set out in 3.2 is the following:

\[
\hat{x}_t = G_t E(y_{t-1} | I_{t-1}) + g_t
\]

where

\[
\begin{align*}
G_t &= - (C_t H_t C_t)^{-1} C_t H_t A_t \\
g_t &= - (C_t H_t C_t)^{-1} C_t (H_t b^*_{t-1} - h^*_{t-1})
\end{align*}
\]

being

\[
\begin{align*}
H_{t-1} &= K_{t-1} + (\tilde{A}_t + \tilde{C}_t G_t)^{H} (\tilde{A}_t + \tilde{C}_t G_t) \\
h_{t-1} &= K_{t-1} a_{t-1} + (\tilde{A}_t + \tilde{C}_t G_t) (h^*_{t-1} - H_t b^*_{t-1}), \quad \text{with} \quad H_T = K_T \\
h_T &= K_T a_T
\end{align*}
\]

The proof of this theorem appears in Cerdá (1987).

### 3.4. Commentaries on the solution obtained

Let us consider the problem of standard linear-quadratic control, with complete information (problem SIC):

\[
\min E_0 W = E_0 \sum_{t=1}^{T} (y_t - a_t)^{H} K_t (y_t - a_t), \quad K_t \text{ being a positive semidefinite, symmetric matrix.}
\]

\[
y_t = \tilde{A}_t y_{t-1} + \tilde{C}_t x_t + \tilde{b}_t + u_t, \quad \text{for } t = 1, ..., T
\]

where the variables \( b_t \) are stochastic and of the form previously expressed.

The optimal solution of this problem is \( \hat{x}_t = G_t y_{t-1} + g_t \), where \( G_t \) and \( g_t \) and the corresponding \( H_t \) and \( h_t \) are given by the very same expressions in theorem 1.

The incomplete information version of the same problem (problem SII)

\[
\min E_0 W = E_0 \sum_{t=1}^{T} (y_t - a_t)^{H} K_t (y_t - a_t), \quad K_t \text{ being a positive semidefinite, symmetric matrix.}
\]
\[ y_t = \bar{A}_t y_{t-1} + \bar{C}_t x_t + \bar{b}_t + u_t, \quad \text{for } t = 1, \ldots, T \]
\[ z_t = M_t y_t + w_t \quad \text{for } t = 0, \ldots, T \]

has as a solution \( \hat{x}_t = G_t E(y_{t-1}|I_{t-1}) + g_t, \) \( G_t, g_t \) being analogous to those obtained in theorem 1, the same as \( H_t, h_t \) and, therefore, also analogous to those of problem SIC. Consequently, this problem and the so-called previous problem have exactly the same solution. Moreover, in the expression of the optimal control of the previous problem, and thus, of problem SII, we find exactly the expression of the optimal control of problem SIC, with the sole change of \( E(y_{t-1}|I_{t-1}) \) in place of \( y_{t-1} \).

The complete developments of the optimal solutions of the problems SIC and SII, with notations and assumptions used in this article appear in Cerdà (1987).

4. SOLUTION TO THE PROBLEM

The following theorem gives us the solution to the problem we are dealing with.

Theorem 2

Let us consider the problem set out in section 2. We assume that, for \( t = T \) (final period), it is verified that \( y_{T+1|T-1}^* = \Gamma y_{T|T-1}^* \).

The optimal solution is \( \hat{x}_t = F_t E(y_{t-1}|I_{t-1}) + f_t \), where \( F_t, f_t \) coincide with the expressions calculated in the case of complete information.

Proof

Our starting point is the version in complete information of the problem we are dealing with and its optimal solution (Cerdà, 1990). For this case, the system can be expressed:

\[ y_t = B_{1t} y_{t+1|t-1}^* + A_t y_{t-1} + C_t x_{t-1}^* + b_{t|t-1}^* + v_t \]

where \( B_{1t}, A_t, C_t, b_{t|t-1}^* \), \( v_t \) coincide with the expressions appearing in proposition 1.

In point 2 of the method used to solve the problem in the case of complete information, \( y_{t+1|t-1}^* \) is treated as given and using dynamic
programming, we obtain:

\[ \hat{x}_{t+1/\eta -1}^* = G_t y_{t-1} + G_{1t} y_{t+1/\eta -1} + g_t \]

In the case we are dealing with now (incomplete information), we have seen in proposition 1 that the system can be expressed as:

\[ y_t = B_t y_{t+1/\eta -1} + A_t y_{t-1} + (\hat{A}_t - A_t) E(y_{t-1} | I_{t-1}) + C_t x_{t+1/\eta -1}^* + \delta_{t+1/\eta -1}^* + v_t \]  \(2\)

We now treat \( y_{t+1/\eta -1}^* \) as given and use theorem 1 along with the results commented on in 3.4, obtaining:

\[ \hat{x}_{t+1/\eta -1} = G_t E(y_{t-1} | I_{t-1}) + G_{1t} y_{t+1/\eta -1}^* + g_t \]  \(3\)

where, \( G_{1t}, G_t, g_t \) coincide with the expressions obtained in the case of complete information.

Taking this result to (2), we obtain:

\[ y_t = R_{1t} y_{t+1/\eta -1}^* + A_t y_{t-1} + (R_t - A_t) E(y_{t-1} | I_{t-1}) + r_t + v_t \]  \(4\)

where

\[ R_{1t} = B_{1t} + C_{t} G_{1t}; \quad R_t = \hat{A}_t + C_t G_t; \quad r_t = \delta_{t+1/\eta -1}^* + C_t g_t \]

(it should be observed that \( r_t = r_{t+1/\eta -1}^* \), but in general, \( r_t \neq r_{t+1/\eta -j}^* \) for \( j > 1 \)).

We are going to solve this system. We will prove by induction that system (4) can be expressed as:

\[ y_t = A_t y_{t-1} + (P_t - A_t) E(y_{t-1} | I_{t-1}) + s_t + v_t \]  \(5\)

where \( P_t = (I - R_{1t} P_{t+1})^{-1} R_t \) for \( t = 1, \ldots, T \) with \( P_{T+1} = \Gamma \)

\[ s_t = (I - R_{1t} P_{t+1})^{-1} (r_t + R_{1t} \delta_{t+1/\eta -1}^*) \]

for \( t = 1, \ldots, T \), with \( s_{T+1} = 0 \).

- For \( T \)

By particularising (4) for \( t = T \), and bearing in mind that \( y_{T+1/\eta -1}^* = \Gamma y_{T/\eta -1} \), we get:

\[ y_T = R_{1T} \Gamma y_{T/\eta -1} + A_T y_{T-1} + (R_T - A_T) E(y_{T-1} | I_{T-1}) + r_T + v_T \]

Taking expectations conditioned to \( I_{T-1} \) on the two sides of this
equation and bearing in mind that expectations are rational, we obtain:

\[ y_{T/T-1}^\pi = R_{1T} \Gamma y_{T/T-1}^\pi + R_T E(y_{T-1}|I_{T-1}) + r_T \]

Then:

\[ y_T = y_{T/T-1}^\pi + A_T y_{T-1} - A_T E(y_{T-1}|I_{T-1}) + v_T = \]
\[ = A_T y_{T-1} + (P_T - A_T) E(y_{T-1}|I_{T-1}) + s_T + v_T. \]

where:

\[ P_T = (I - R_{1T} \Gamma)^{-1} R_T \]
\[ s_T = (I - R_{1T} \Gamma)^{-1} r_T \]

- Let us assume that (5) is true for \( t + 1 \). Then

\[ y_{t+1} = A_{t+1} y_t + (P_{t+1} - A_{t+1}) E(y_{t+1}|I_t) + s_{t+1} + v_{t+1} \]

We are going to prove it for \( t \).

We start from equation (4).

From the starting point of the induction hypothesis for \( t + 1 \), and taking expectations conditioned to \( I_{t-1} \) we get:

\[ y_{t+1/i_{t-1}}^* = P_{t+1} y_{t+1/i_{t-1}}^* + s_{t+1/i_{t-1}}^* \]

Taking this result to (4) and, taking expectations conditioned to \( I_{t-1} \), we obtain:

\[ y_{t+1/i_{t-1}}^* = R_{1t} P_{t+1} y_{t+1/i_{t-1}}^* + R_{1t} s_{t+1/i_{t-1}}^* + A_t E(y_{t-1}|I_{t-1}) + \]
\[ + (R_t - A_t) E(y_{t-1}|I_{t-1}) + r_t \]

Then:

\[ y_t = y_{t+1/i_{t-1}}^* + A_t y_{t-1} - A_t E(y_{t-1}|I_{t-1}) + v_t = \]
\[ = A_t y_{t-1} + (P_t - A_t) E(y_{t-1}|I_{t-1}) + s_t + v_t \]

where

\[ P_t = (I - R_{1t} P_{t+1})^{-1} R_t \]
\[ s_t = (I - R_{1t} P_{t+1})^{-1} (r_t + R_{1t} y_{t+1/i_{t-1}}^*) \]

This is the solution of the system we were looking for.
The values of $P_n$ that we have obtained coincide with those calculated in the case of complete information. We have, therefore,

$$y_t = A_t y_{t-1} + (P_t - A_t)E(y_{t-1} I_{t-1}) + s_t + v_t$$

$$\Rightarrow y_{t+1} = A_{t+1} y_t + (P_{t+1} - A_{t+1})E(y_t I_t) + s_{t+1} + v_{t+1}$$

$$\Rightarrow y_{t+1}^* = P_{t+1} y_{t+1}^* + s_{t+1}^*$$

From (6) we obtain

$$y_{t+1}^* = P_t E(y_{t-1} I_{t-1}) + s_t$$

Therefore,

$$y_{t+1}^* = P_{t+1} P_t E(y_{t-1} I_{t-1}) + P_{t+1} s_t + s_{t+1}^*$$

Taking this result to (3) we have

$$\hat{x}_{t+1}^* = (G_t + G_{1t} P_{t+1} P_t) E(y_{t-1} I_{t-1}) + g_t + G_{1t} P_{t+1} s_t + G_{1t} s_{t+1}^*$$

Therefore,

$$\hat{x}_{t+1}^* = \hat{x}_t + F_t E(y_{t-1} I_{t-1}) + f_t$$

where

$$F_t = G_t + G_{1t} P_{t+1} P_t$$

$$f_t = g_t + G_{1t} (P_{t+1} s_t + s_{t+1}^*)$$

We thus see that $F_t, f_t$ coincide with the expressions obtained in the case of complete information and the theorem is proved.

Note: In the proof of the theorem we have assumed that the exogenous variables $\{b_t\}$ are stochastic and of the type

$$b_t = \sum_{i=1}^{p} R_i b_{t-i} + \xi_t$$

The theorem is equally true for the case of deterministic variables $\{b_t\}$ and the proof is analogous.
5. CONCLUSIONS

In the above pages the problem of control of linear systems with quadratic objective function and finite time horizon, for the case of incomplete information, has been set out and solved. In Cerdá (1990) the corresponding case of complete information is solved.

Although the usual Control Theory techniques are not applicable to his case since they do not meet the causality hypothesis, the final result which is obtained preserves the property known in standard linear-quadratic problems: the expression of the optimal control for the case of incomplete information exactly coincides with that obtained in complete information, with the sole change being $y_{t-1}$ for $E(y_{t-1}|I_{t-1})$.

In the Gaussian case this conditional expectation $E(y_{t-1}|I_{t-1})$ will be calculated by using the Kalman’s filter for the state equation and the observation system.

REFERENCES


