DUALITY THEOREMS FOR A CLASS OF
NON-LINEAR PROGRAMMING PROBLEMS

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RESUMEN

La dualidad de la programación lineal se usa para establecer un importante
teorema de dualidad para una clase de problemas de programación no-lineal.
El problema primario tiene una función objetiva cuasimonotónica y un polié-
dro convexo como su limitación (constraint set).

Palabras clave: dualidad, programación no-lineal, función cuasimonotónica.

Clasificación AMS: 90C30 (programación no-lineal).

Title: Duality Theorems for a Class of Non-Linear Programming Problems

SUMMARY

Duality of linear programming is used to establish an important duality
theorem for a class of non-linear programming problems. Primal problem has
quasimonotonic objective function and a convex polyhedron as its constraint
set.

Key words: duality, non-linear programming, quasimonotonic function.

AMS classification: 90C30 (non-linear programming).

1. INTRODUCTION

The concept of duality is investigated briefly for a class of nonlinear programming problems. We are interested in a nonlinear programming problem whose quasimonotonic objective function needs to be optimized over a constraint set formed by linear inequalities. Results proved here generalize the duality results of Kaska (1969). The following minimization problem is taken as the primal problem (P-P).

\[
\begin{align*}
\text{Minimize } & f(x) \\
\text{Subject to } & x \in S \\
\text{Where } & S = \{x : Ax \geq b, x \geq 0\}
\end{align*}
\]

The nonlinear function \( f(x) \) is quasimonotonic over the set \( S \); \( A = (a_1, a_2, \ldots, a_m) \) is an \( m \times n \) matrix; \( x \) and \( b \) are column vectors with \( n \) and \( m \) components respectively.

Before we formulate a dual of problem (1), we state and prove a lemma.

Lemma 1

A basic feasible solution \( x_0 = (x'_b, 0)' \) is an optimal solution to problem (1) if

\[
f_{x_0}(x_0) - Y_j f_{a_b}(x_0) \geq 0
\]

for all columns \( a_j \) of the matrix \((A, I)\) in \( S \). \( f_{x}(x) \) in the \( n \times 1 \) gradient vector of \( f(x) \) at \( x' \) over a matrix is used to denote its transpose. \( Y_j = B^{-1}a_j \), where \( B \) is the basis matrix corresponding to the basic feasible solution \( x_0 \).

Proof:

\( f(x) \) is quasimonotonic over the set \( S \). It follows from Martos (1965) that \( f(x) \) will attain its minimum at an extreme point of the set \( S \). Let \( x = (x'_b, 0)' \) be a basic feasible solution of problem (1).

In the case when \( x \) does not minimize \( f(x) \), then an improved value of \( f(x) \) (assuming non-degeneracy) can be obtained by inserting some
column $a_j$ into the basis and by deleting column $a_k$ from the basis. The
new basic feasible solution, $\hat{x}$ is computed from Hadley (1963)

$$\hat{x} = x_B - \theta Y_j$$

$$x_j = \frac{x_{Bk}}{y_{kj}} = \theta$$

where

$$Y_j = B^{-1}a_j = (y_1, y_2, ... , y_m)'$$

Inserting the 0-components corresponding to the non-basic variables,
above can be written as

$$\hat{x} = x - \theta \overline{Y}_j + \theta e_j$$

here

$$\overline{Y}_j = (Y_j', 0)'$$

$f(x)$ is quasimonotonic, therefore from Martos (1965) it follows that
$f(\hat{x}) \geq f(x)$ implies that

$$(\hat{x} - x)'f_x(x) = \theta(e_j - \overline{Y}_j)f_x(x)$$

$$= \theta[f_x(\hat{x}) - Y_j'y_{BA}(\hat{x})] \geq 0.$$ 

Under the non-degeneracy assumptions, if for some $j$,

$$f_x(x) - Y_j'y_{BA}(\hat{x}) < 0$$

then $f(x) > f(\hat{x})$. This shows that the insertion of the $a_j$-th column
into the basis decreases the value of the objective function. Thus if $x_0$
$= (x_{B'}, 0)'$ is a minimizing solution of the problem (1), then

$$f_x(x_0) - Y_j'y_{BA}(x_0) \geq 0$$

should hold true for all $j's = 1, 2, ..., n$. This proves the lemma.
The dual problem, (D-P), is now defined as

Maximize \( F(u, v) = f(u) \)

Subject to \( A'v - f_x(u) \leq 0 \) \hspace{1cm} (3)
\( -b'v + u'[f_x(u)] \leq 0 \) \hspace{1cm} (4)
\( u, v \geq 0 \)

Let constraint set of (D-P) be denoted by \( T \).

2. THE DUALITY THEOREMS

Theorem 1:

Let \( G \) be the infimum of \( f(x) \) over \( S \), and \( g \) be the supremum of \( F(u, v) \) over \( T \), then

\[ g \leq G \]

Proof:

We adopt the convention that \( G = +\infty \) if the set \( T \) is null and that \( g = -\infty \) if the set \( S \) is null. It is, therefore, sufficient to prove theorem for the case when \( T \) and \( S \) are non-null.

Let \( x \in S \) and \( (u, v) \in T \). From (3) it follows that

\[ v'Ax \leq [f_x(u)]'x \] \hspace{1cm} (5)

Also from \( Ax \geq b \), we have, \( v'Ax \geq v'b \). This fact along with (4) when used in (5) yields

\[ (x - u)'f_x(u) \geq 0 \] \hspace{1cm} (6)

\( f(x) \) is quasimonotonic, therefore, from (6) it follows that

\[ f(x) \geq f(u) = F(u, v) \]
Theorem 2:

Let $x_0$ be a feasible solution for (P-P), and $(u_0, v_0)$ be a feasible solution for (D-P), such that

$$f(x_0) = F(u_0, v_0).$$

Then $x_0$ will minimize $f(x)$ and $(u_0, v_0)$ will maximize $F(u, v)$.

Proof:

From theorem 1, for any $x \in S$, we have

$$f(x) \geq F(u_0, v_0) = f(x_0).$$

This shows that $x_0$ will minimize $f(x)$. Again, as for any $(u, v) \in T$, we have

$$F(u, v) \leq f(x_0) = F(u_0, v_0),$$

therefore, $(u_0, v_0)$ maximizes $F(u, v)$.

Lemma 2:

Let $x_0$ be a solution to the primal problem. Then $x_0$ will also be a solution to the following linear programming problem.

Minimize \( [f(x_0)]'x \)

Subject to $x \in S$ \hfill (7)

Proof:

Let $x_0$ minimize (7). Then, we must have $z_j - c_j \leq 0$, for all $j$'s $= 1, 2, ..., n$ (Hadley, 1963). In the present context, optimality condition of linear programming reduces to

$$f_{x_j}(x_0) - Y_jf_{x_j}(x_0) \geq 0$$

This is same as (2), the condition for $x_0$ to minimize the primal problem. This proves the lemma.

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Theorem 3:

If \( x_0 \) solves the primal problem, then there exists \( v_0 \) so that \((x_0, v_0)\) solves the dual problem, and the extrema are equal.

Proof:

From above lemma, we have seen that if \( x_0 \) minimizes (P-P), then \( x_0 \) is a solution of (7) also. Further, dual of linear programming problem (7) is to

Maximize \( b'v \)
Subject to \( A'v \leq f_d(x_0) \)
\[ v \geq 0 \]

(8)

Duality in linear programming ensures existence of optimal \( v_0 \) for problem (8) and that the optimal values of objective functions in both the cases are equal i.e.

\[
b'v_0 = [f_d(x_0)]' \cdot x_0
\]

(9)

From (8) and (9) it is clear that \((x_0, v_0)\) is a feasible solution for the dual problem. The fact that \((x_0, v_0)\) is optimal for the (D-P) is evident from Theorems 1 and 2. Moreover, optimal values of objective functions in both the cases are equal.

3. DUALITY IN LINEAR FRACTIONAL PROGRAMMING

Let our primal problem be

Minimize \( f(x) = \frac{c x + c_0}{d x + d_0} = \frac{G(x)}{H(x)} \)
Subject to \( A x \geq b \)
\[ x \geq 0 \]

(10)
Here $c$ and $d$ are row vectors with $n$ components and $c_0, d_0$ are arbitrary constants. The objective function $f(x)$ is quasimonotonic in nature. The dual of (10) will be to

\[
\text{Maximize } F(u, v) = \frac{cu + c_0}{du + d_0} = \frac{G(u)}{H(u)} = f(u)
\]

Subject to $A'v + \frac{d}{H(u)} f(u) \leq \frac{c}{H(u)}$

and $-b'v + \frac{cu}{H(u)} \leq \frac{du}{H(u)} f(u)$

\[u, v \geq 0\] (11)

In (11), let us take $f(u) = t$, $cu = G(u) - c_0$, and $du = H(u) - d_0$. Dual of (10) becomes

\[
\text{Maximize } t
\]

Subject to $A'v + \frac{d}{H(u)} t \leq \frac{c}{H(u)}$

and $-b'v + \frac{du}{H(u)} t \leq \frac{c_0}{H(u)}$

\[u, v \geq 0\] (12)

(10) and (12) are respectively the primal and dual problems of Kaska (1969).

REFERENCES


