ON EXACT CONDITIONALS

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1. Introduction.

1.1. Let it be \( E \) a Boolean Algebra of propositions \( a, b, c, \ldots \) of which those belonging to a previous given subset \( T \subset E \) are true and the others in \( F = E - T \) are false.

When, in Commonsense Reasoning, it is affirmed a conditional relation

"If \( a \), then \( b \)"

(for short \( a \Rightarrow b \)) it is also affirmed that \( a' \cdot (b + b') + ab = a' + b \in T \), provided that boolean operations \( + \) of join, of meet and \( ' \) of negation verify the properties: \( a \in T \iff a' \in F; \)

\( a \cdot b \in T \iff a \in T \) and \( b \in T \), and \( a + b \in T \iff a \in T \) or \( b \in T \).

Then, it is supposed that the relation of Material Conditional associated with \( T \):

\[
\begin{align*}
a \rightarrow_T b & \iff a' + b \in T,
\end{align*}
\]

contains \( \Rightarrow \subset \rightarrow_T \) \([4]\).

The frequently made hypothesis \( \Rightarrow \rightarrow_T \) conveys the undesired consequence of \( a \Rightarrow b \), if \( a \in F \). That fact, important in Formal Reasoning, is not usual in Commonsense Reasoning [1]. It is rare to affirm as a piece of Commonsense Reasoning, something like

"If Madrid is the capital of France, then this is a paper on Logic".

no matter if "this is a paper on Logic" is true or false.

What is actually supposed satisfies a conditional relation is only the so-called Modus

Ponens Rules:

If \( a \in T \) and \( a \Rightarrow b \), then \( b \in F \),
that implies the **Modus Tollens Rule**:

\[
    \text{If} \quad b \in F \quad \text{and} \quad a \Rightarrow b, \quad \text{then} \quad a \in F,
\]

**Definition 1.1.1.** Given a set \( E \) and a non-empty subset \( T \subset E \), a binary relation on \( E \), \( \Rightarrow \subset E \times E \), is a \( T \)-conditional on \( E \) if:

\[
    a \in T \quad \text{and} \quad a \Rightarrow b \quad \text{implies} \quad b \in T.
\]

It is clear that if \( \Rightarrow \) is a \( T \)-conditional on \( E \) and \( \Rightarrow \supset \) is another relation on \( E \) such that \( \Rightarrow \supset \Rightarrow \Rightarrow \), then \( \Rightarrow \Rightarrow \) is a \( T \)-conditional on \( E \). Sometimes, when 1.1.1. holds, it is said that \( T \) is a \( t \)-set (t for true) or a **Logic State** for the relational structure \( (E, \Rightarrow) \).

If \( T = \{ 1 \} \), as \( a \Rightarrow_1 b \iff a' + b = 1 \iff a \leq b \), it is clear that the \{1\}-Material Conditional in a Boolean Algebra is its partial order. \( 1 \) is the largest element of the Boolean Algebra \( E \).

In what follows we will study such kind of exact relations, \( T \)-conditionals (of which \( T \)-Material Conditional is the best known) without using any algebraic structure on the ground proposition’s set \( E \). We speak of exact as more restrictive than inexact, in the sense of [2] and [4].

1.2 To chain pieces of reasoning it is convenient that a relation \( \Rightarrow \) on \( E \), modelizing a conditional, be transitive. But if it is not the case we can extend \( \Rightarrow \) to its transitive clausure \( \Rightarrow_\tau \); \( a \Rightarrow_\tau b \) means that \( a \Rightarrow a_1, a_1 \Rightarrow a_2, \ldots, a_n \Rightarrow b \), for some propositions \( a_1, \ldots, a_n \) in \( E \). It should be realized that \( \Rightarrow \subseteq \Rightarrow_\tau \).

**Theorem 1.2.1.** A relation \( \Rightarrow \) verifies the Rule of **Modus Ponens** if and only if \( \Rightarrow_\tau \) does.

**Proof.** If \( \Rightarrow_\tau \) is a \( T \)-conditional, \( \Rightarrow \) is a \( T \)-conditional. Reciprocally, if \( a \in T \) and \( a \Rightarrow_\tau b \), is \( a \in T \) and \((a \Rightarrow a_1, a_1 \Rightarrow a_2, \ldots, a_n \Rightarrow b) \) or \( a \in T \), and \( a_1 \in T \), and \( a_2 \in T \), \ldots, and \( a_n \in T \) and \( b \in T \).
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It is frequently supposed that a T-conditional satisfies the weak condition of reflexivity: \( a \Rightarrow a \), for each \( a \in E \), translating the usual affirmation "If \( a \), then \( a \)". If relation \( \Rightarrow \) is not reflexive, it can be extended to its reflexive clauseure

\[ \Rightarrow_r := \Rightarrow \cup \{(a, a); a \in E\} \].

Of course \( \Rightarrow \subseteq \Rightarrow_r \).

**Theorem 1.2.2.** A relation \( \Rightarrow \) is a T-conditional if and only if \( \Rightarrow_r \) is a T-conditional.

**Proof.** If \( \Rightarrow_r \) is a T-conditional is obvious that \( \Rightarrow \) does. Reciprocally, if \( a \in T \) and \( a \Rightarrow_r b \), it is \( a = b \) (and \( b \in T \)) or \( a \neq b \) and then \( a \Rightarrow b \) and \( b \in T \).

If \( \Rightarrow \) is not transitive and reflexive, we can proceed from \( \Rightarrow \) to \( \Rightarrow_{rt} \):

\[ \Rightarrow_{rt} \subseteq \Rightarrow_r \subseteq \Rightarrow_{rt} \],

and \( \Rightarrow \) is a T-conditional if \( \Rightarrow_{rt} \) is a T-conditional.

2. T-conditionals.

Next result shows an intrinsic representation of the Material Conditional.

**Theorem 2.1.** Given \((E, T)\), the relation \( \rightarrow_T = (F \times E) \cup (T \times T) \) is the greatest T-conditional.

**Proof.** Let's consider the set \( C_T = \{ \Rightarrow \subseteq E \times E; \Rightarrow \text{ is a T-conditional} \} \); that set is non-empty, for example \( T \times T \) belongs to \( C_T \). Consider

\[ \rightarrow_T = \bigcup_{\Rightarrow \in C_T} \Rightarrow \]
Such relation is a T-conditional: if $a \rightarrow_T b$, it should be also $a \Rightarrow b$ for some $\Rightarrow \in C_T$, and then if $a \in T$ it is $b \in T$. Obviously $\rightarrow_T$ is the greatest T-conditional.

If $a \in T$, for having $a \rightarrow_T b$ for some $b \in E$, it should be $b \in T$. But if $a \in F$, it is always $a \rightarrow_T b$ for any $b \in E$, because $\Rightarrow = T \times T \cup \{(a,b)\}$ is a T-conditional such that $a \Rightarrow b$. Then $\rightarrow_T = (F \times E) \cup (T \times T)$.

**Corollary.** A relation $\Rightarrow \subseteq E \times E$ is a T-conditional if and only if $\Rightarrow \subseteq \rightarrow_T$.

**Proof.** By theorem 2.1 if $\Rightarrow$ is a T-conditional, then $\Rightarrow \subseteq \rightarrow_T$. Reciprocally, if $a \in T$ and $a \Rightarrow b$ it is $a \in T$ and $a \rightarrow_T b$ and, being $\rightarrow_T$ a T-conditional, $b \in T$.

**Theorem 2.3.** The T-Material Conditional is a Preorder.

**Proof.** For $a \in E$, it is $a \in T$ and $a \rightarrow_T a$, or it is $a \in F$ and, as $a \in E$, it is also $a \rightarrow_T a$.

Suppose $a \rightarrow_T b$ and $b \rightarrow_T c$. If $a \in F$, as $c \in E$, it is $a \rightarrow_T c$; if $a \in T$, then $b \in T$ and $c \in T$, and $a \rightarrow_T c$.

**Corollary.** Given a set $A \subseteq E$, the relation $\rightarrow_A = (E - A) \times E \cup A \times A$ is a preorder, the preorder by $A$.

If $\Rightarrow$ is a T-conditional such that when $a \in F$ it is $a \Rightarrow b$ for any $b \in E$, then $F \times E \subseteq \Rightarrow \subseteq \rightarrow_T$.

If $\{1\} \subseteq T$ it is $\rightarrow_{\{1\}} \subseteq \rightarrow_T$ and, in that restricted sense of monotonicity, the classical material conditional $\rightarrow_{\{1\}} \leq$ is the more conservative: every conditional $a \leq b$ implies the conditional $a \rightarrow_T b$, for any set $T$ containing 1.

3. On consequences and conditionals.

Let’s consider for any relation $\Rightarrow \subseteq E \times E$ the mapping $C_\Rightarrow : P(E) - \{\emptyset\} \rightarrow P(E) - \{\emptyset\}$,
given by [3]:

\[ C_{\Rightarrow}(T) = \{ x \in E; \exists a \in T : a \Rightarrow x \}, \]

for each \( T \subseteq E, T \neq \emptyset \). It is obvious that \( C_{\Rightarrow} \) is monotone: if \( A \subseteq B \) then \( C_{\Rightarrow}(A) \subseteq C_{\Rightarrow}(B) \). It is also obvious that \( \Rightarrow_1 \subseteq \Rightarrow_2 \) implies \( C_{\Rightarrow_1}(A) \subseteq C_{\Rightarrow_2}(A) \).

**Theorem 3.1.** Relation \( \Rightarrow \) is a T-conditional, for \( \emptyset \neq T \subseteq E \), if and only if \( C_{\Rightarrow}(T) \subseteq T \).

**Proof.** If \( C_{\Rightarrow}(T) \subseteq T \), then if \( a \in T \) and \( a \Rightarrow b \), as \( b \in C_{\Rightarrow}(T) \), it is \( b \in T \), and \( \Rightarrow \) is T-conditional. Reciprocally, if \( \Rightarrow \) is a T-conditional and \( x \in C_{\Rightarrow}(T) \), as \( a \Rightarrow x \) for some \( a \in T \), it is \( x \in T \).

It should be pointed out that, if \( T \) is finite, \( C_{\Rightarrow}(T) \) should not be also finite. Just consider \( E = \mathbb{N}, \Rightarrow = \mathbb{N} \times \mathbb{N} \) and \( T = \{ 1 \} \): it is \( C_{\Rightarrow}(T) = \mathbb{N} \). Nevertheless, being \( E \) finite or \( \Rightarrow \) finite, if \( T \) is finite so it is \( C_{\Rightarrow}(T) \).

**Theorem 3.2.** A relation \( \Rightarrow \) is reflexive if and only if \( T \subseteq C_{\Rightarrow}(T) \) for any \( \emptyset \neq T \subseteq E \).

**Proof.** If \( \Rightarrow \) is reflexive, as \( a \Rightarrow a \) for each \( a \in T \), it is \( a \in C_{\Rightarrow}(T) \) and \( T \subseteq C_{\Rightarrow}(T) \). Reciprocally, for any \( a \in E \) it is \( \{ a \} \subseteq C_{\Rightarrow}(\{a\}) \), and \( a \Rightarrow a \).

**Corollary.** A reflexive relation \( \Rightarrow \) is a T-conditional iff \( T = C_{\Rightarrow}(T) \).

**Theorem 3.3.** A relation \( \Rightarrow \) is transitive if and only if \( C_{\Rightarrow}(C_{\Rightarrow}(T)) \subseteq C_{\Rightarrow}(T) \), for any non-empty subset \( T \) of \( E \).

**Proof.** If \( a \Rightarrow b \) and \( b \Rightarrow c \), from \( b \in C_{\Rightarrow}(\{a\}) \) and \( c \in C_{\Rightarrow}(\{b\}) \) it follows \( c \in C_{\Rightarrow}(\{b\}) \subseteq C_{\Rightarrow}(C_{\Rightarrow}(\{a\})) \), and \( a \Rightarrow c \). Reciprocally, being \( \Rightarrow \) transitive, if \( x \in C_{\Rightarrow}(C_{\Rightarrow}(T)) \) it exists some \( b \in C_{\Rightarrow}(T) \) such that \( b \Rightarrow x \); but it also exists some \( c \in T \) such that \( c \Rightarrow b \): then \( c \Rightarrow x \), or \( x \in C_{\Rightarrow}(T) \).
Corollary. If \( \Rightarrow \) is transitive, it is a \( C_{\Rightarrow}(T) \)-conditional for any \( \emptyset \neq T \subset E \).

Corollary. If \( \Rightarrow \) is a preorder, it is a \( C_{\Rightarrow}(T) \)-conditional and \( T \subset C_{\Rightarrow}(T) \) for any \( \emptyset \neq T \subset E \).

Corollary. A reflexive relation \( \Rightarrow \) is transitive iff \( C_{\Rightarrow}(C_{\Rightarrow}(T)) = C_{\Rightarrow}(T) \) for each \( T \subset E \), \( T \neq \emptyset \).

Theorem 3.4. Mapping \( C_{\Rightarrow} \) is a Tarski’s Consequences Operator [3] iff \( \Rightarrow \) is a preorder.

Proof. Is an immediate consequence of theorem 3.2 and 3.3. Then, being \( \Rightarrow \) a preorder, it has complete sense to say that \( b \) is a consequence of \( a \), each time that \( a \Rightarrow b \).

Theorem 3.5. If \( \Rightarrow \) is a preorder, for any \( \emptyset \neq T \subset E \), it is \( C_{\Rightarrow}(T) \) the smallest subset of \( E \) that contains \( T \) and for which \( \Rightarrow \) is a conditional.

Proof. The set \( \mathcal{C} = \{X \subset E; T \subset X \text{ and } \Rightarrow \text{ is and } X \text{-conditional}\} \) is not-empty because \( E \in \mathcal{C} \). Let it be

\[
\overline{T} = \bigcap_{C \in \mathcal{C}} X.
\]

It is \( T \subset \overline{T} \); then \( C_{\Rightarrow}(T) \subset C_{\Rightarrow}(\overline{T}) \). It is \( \overline{T} \subset C_{\Rightarrow}(\overline{T}) \); if \( x \in C_{\Rightarrow}(\overline{T}) \) it exists some \( a \in \overline{T} \) such that \( a \Rightarrow x \) and, as \( \Rightarrow \) is and \( \overline{T} \)-conditional, \( x \in \overline{T} \) and \( C_{\Rightarrow}(\overline{T}) \subset \overline{T} \); but as \( \Rightarrow \) is a \( C_{\Rightarrow}(T) \)-conditional it is \( \overline{T} \subset C_{\Rightarrow}(T) \) and, finally, \( \overline{T} = C_{\Rightarrow}(T) \).

Corollary. Given a preorder \( \Rightarrow \) on \( E \) and a subset \( T \subset E \), \( T \neq \emptyset \), it suffices to extend \( T \) to \( C_{\Rightarrow}(T) \) for having that \( \Rightarrow \) is a \( C_{\Rightarrow}(T) \)-conditional, provided that \( \Rightarrow \) does not to be a \( T \)-conditional.

Then, each time that \( a \Rightarrow b \) for both \( a \) and \( b \) in \( T \), we can say that \( b \) is a consequence of \( a \). It should be remarked that, if \( \Rightarrow \) is not a preorder it can be extended to the preorder \( \Rightarrow_{rt} \) for which follows the last assertion. In any case, if \( \Rightarrow \) is not a preorder, but it is a
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$T$-conditional, as $\Rightarrow \subset T$, it follows

\[ C_{\Rightarrow}(T) \subset C_{\Rightarrow A}(T), \]

and each $x \in C_{\Rightarrow}(T)$ can be considered as a consequence of $T$.

**Theorem 3.6.** Given $(E, \Rightarrow)$ and a function $\mu : E \to [0,1]$ such that "If $a \Rightarrow b$, then $\mu(a) \leq \mu(b)$", then, for each $\epsilon \in (0,1]$, is $\Rightarrow a\mu^{-1}([\epsilon,1])$-conditional.

**Proof.** If $a \in \mu^{-1}([\epsilon,1])$ and $a \Rightarrow b$, it is $\epsilon \leq \mu(a) \leq 1$ and $\mu(a) \leq \mu(b) \leq 1$, then $\epsilon \leq \mu(b) \leq 1$ and $b \in \mu^{-1}([\epsilon,1])$.

For example, if $E$ is a Boolean Algebra and $p$ is a probability on $E$, as $a \leq b$ implies $p(a) \leq p(b)$, the partial order $\leq$ is a $P_\epsilon$-conditional, being

\[ P_\epsilon = \{ x \in E : \epsilon \leq p(x) \leq 1 \}, \]

for each $\epsilon$ in $(0,1]$.

The last theorem opens the door to exactify some parts of Approximate Reasoning [5].

**References.**


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