ON INARIANT MEASURES FOR THE TEND MAP

F. BOFILL

ABSTRACT

The bifurcation structure of a one parameter dependent piecewise linear population model is described. An explicit formula is given for the density of the unique invariant absolutely continuous probability measure \( \mu_\beta \) for each parameter value \( \beta \). The continuity of the map \( \beta \to \mu_\beta \) is established.

1. We consider a piecewise linear simplification of the logistic model, given by the dynamical system (Fig. 1a)

\[
f_\beta(x) = \begin{cases} 
\beta x & \text{if } x \in [0,1/2] \\
\beta(1-x) & \text{if } x \in [1/2,1], \quad \beta \in (1,2).
\end{cases}
\]

Every point \( x \in (0,1) \) enters under iteration the interval \([f_\beta^2(1/2), f_\beta(1/2)]\) and keeps in it. The dynamics of the system has its interest in this interval. If we restrict \( f_\beta \) to it and rescale to \([0,1]\) we obtain the function (Fig. 1b)

\[
F_\beta(x) = \begin{cases} 
\beta(x-1) + 2 & \text{if } x \in [0,c] \\
\beta(-x+1) & \text{if } x \in [c,1], \quad c = 1 - 1/\beta.
\end{cases}
\]

\( F_\beta(x) \) has a fixed point at \( x^* = \beta/(\beta + 1) \).

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The variation of the parameter $\beta$ gives rise to a period doubling bifurcation structure where the attractive period is constituted by intervals.

More precisely the bifurcation appears for the values $\beta = 2^{\frac{1}{2}}, 2^{\frac{3}{4}}, \ldots, 2^{\frac{k}{2}}, \ldots$.

If $2^{\frac{1}{2}} \leq \beta \leq 2$ (which corresponds to $F_\beta(0) \leq x^*$) the set of points whose orbit under $F_\beta$ is not dense in $[0,1]$ has null Lebesgue measure. In particular $F_\beta$ is transitive.

For $1 < \beta < 2^{\frac{1}{2}}$ ($F_\beta(0) > x^*$) the disjoint intervals $I_1 = [0, F_\beta^2(0)]$, $I_2 = [F_\beta(0), 1]$ are mapped onto each other and are invariant under $F_\beta^2 : F_\beta^2 I_2$ is conjugated to $F_{\beta'}$ on $[0,1]$.

Every $x \in [0,1]$, $x \neq x^*$ is attracted under $F_\beta$ to $I_1, I_2$.

Inductively one can show that in the range $2^{\frac{1}{2^{2^n}}} \leq \beta < 2^\frac{1}{2^n}$, $n \geq 1$, there exists an $F_\beta$-attractive periodic chain formed by $2^n$ disjoint closed $F_{\beta'}^{2^n}$-invariant subintervals $I_{i_1, i_2, \ldots, i_n}$ ($i_j \in \{1, 2\}$), in $[0,1]$ and that on $I_{2,2,\ldots,2} = [F_{\beta'}^{2^{2^n}-1}(0), 1]$, $F_{\beta'}^{2^n}$ is conjugated to the transitive function $F_{\beta'}^{2^{\frac{1}{2}}} (2^{\frac{1}{2}} \leq \beta^{2^n} < 2)$, on $[0,1]$. (Figure 2).
The bijection \( v_{2,3,\ldots,2} \rightarrow v = \frac{1}{2^n} \sum_{i=0}^{2^n-1} F^i_\beta \cdot v_{2,3,\ldots,2} \) identifies the probabilities on \( I_{2,3,\ldots,2} \) invariant under \( F^{2^n}_\beta \approx F^{n+1}_\beta \) with the probabilities on \([0,1]\) invariant under \( F_\beta \) vanishing on the fixed points of \( F_\beta, F^{2}_\beta, \ldots, F^{2^n-1}_\beta \), or, equivalently, on \([0,1] - \cup I_{1,\ldots,n}\).

2.- The results in papers [5] and [6], when applied to the model \( F_\beta \), ensure the existence of a unique absolutely continuous \( F_\beta \)-invariant probability measure \( \mu_\beta \).

Take \( s_0 = 1 \) and, respectively, \( s_n = 1,0,-1 \) when \( F^{n+1}_\beta(0) < 1 - \frac{1}{\beta} , F^n_\beta(0) = 1 - \frac{1}{\beta} \), \( F^n_\beta(0) > 1 - \frac{1}{\beta} , n \geq 1 \). Define \( \Delta_0 = 1 \), and, inductively \( \Delta_n = \Delta_{n-1} - 1 \), \( \Delta_n \geq 1 \).

Take \( \rho_\beta(x) = \sum_{n=0}^{\infty} \Delta_n \frac{\lambda}{\beta^n} \chi_{\{F^n_\beta(0),0\}}(x) \) and let \( K_\beta = \int_0^1 \rho_\beta(x)dx \) be a normalizing factor.

\[ \mu_\beta(A) = \frac{1}{K_\beta} \int_0^1 \rho_\beta(x)dx \] is the unique \( F_\beta \)-invariant absolutely continuous probability measure.

If \( 2^{1/\beta} < \beta < 2^{1/\beta} \), \( \rho_\beta(x) > 0 \) for \( x \in I_{1,\ldots,n} \) and \( \rho_\beta(x) = 0 \) for \( x \in [0,1] - \cup I_{1,\ldots,n} \).
We refer to [7], [8] for explicit versions of invariant measures in respect of other related piecewise linear maps on the interval.

3.- Given $\beta \in (1,2]$ let $s_n^\beta$, the sequence defined in 2.- corresponding to $F_\beta$.

If $\beta_0$ is such that $s_n^\beta_0 \neq 0$, $n \geq 1$, the map $\beta \rightarrow \rho_\beta(x)$ from $(1,2]$ to $L^1[0,1]$ is continuous at $\beta = \beta_0$.

Let $\beta_0$ be such that $s_n^\beta_0 = 0$ for some $n$, and let $M > 0$ be the smallest integer with this property. If

$$\omega = \frac{1}{\beta_0^{M+2}} + \frac{1}{(\beta_0^{M+2})^2} + \ldots$$

then $\rho_\beta(x) \rightarrow (1 + \omega)\rho_{\beta_0}(x)$ when $\beta \uparrow \beta_0$ and $\rho_\beta(x) \rightarrow (1 - \omega)\rho_{\beta_0}(x)$ when $\beta \downarrow \beta_0$.

As a consequence one has the following

**Theorem.** The map $\beta \rightarrow \frac{1}{\int \rho_\beta(x)dx}$ is continuous for $\beta \in (1,2]$.

And, if one takes in account the weak topology in the space of measures,

**Corollary.** The map $\beta \rightarrow \mu_\beta$ is continuous.

References.


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E.T.S. d'Enginyers Industrials de Terrassa
Univ. Politècnica de Catalunya
Colón 11, 08222 Terrassa. SPAIN.