A NOTE ON THE ALMOST SURE LIMITING BEHAVIOR
OF THE MAXIMUM OF A SEQUENCE OF PARTIAL SUMS

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ABSTRACT

The goal of this paper is to show that, in most strong laws of large numbers,
the $n^{th}$ partial sum can be replaced with the largest of the first $n$ sums. Moreover,
it is shown that the usual assumptions of independence and common distribution
are unnecessary and that these results also apply to strong laws for Banach valued
random elements.

1. Introduction.

Consider a probability space $(\Omega, \mathcal{F}, P)$ with random variables $(Y_n, n \geq 1)$. Let
$S_n = \sum_{k=1}^{n} Y_k$ and $M_n = \max_{1 \leq k \leq n} S_k$, $n \geq 1$. Let $(b_n, n \geq 1)$ and $(C_n, n \geq 1)$ be our norming
and centering constants, e.g. $(S_n - C_n)/b_n \to 0$ almost surely (a.s.).

We show that strong laws for $S_n$ extend to those for $M_n$. Heyde [5] established
comparable results when the random variables are independent and identically distributed
(i.i.d.).


In this section we establish two theorems which allow us to generalize previous strong
laws of large numbers. The difference in these results is that in one case the centering
sequence is positive and in the other it’s negative. Note that the case of $(C_n, n \geq 1)$
changing sign infinitely often is not of interest, since in most strong laws $C_n/b_n$ approaches a constant. Similarly the monotonicity of $(C_n, n \geq 1)$ in Theorem 1 is not too restrictive.

For, if $S_n/b_n \rightarrow c$ a.s. $(|c| \neq \infty)$, then $C_n = c b_n$, which is nondecreasing in $n$.

**Theorem 1.** If $0 \leq C_n \uparrow \infty$, $0 < b_n \uparrow \infty$, and $\frac{S_n - C_n}{b_n} \rightarrow 0$ a.s., then $\frac{M_n - C_n}{b_n} \rightarrow 0$ a.s.

**Proof.** For each $n \geq 1$ either $M_n \geq C_n$ or $M_n < C_n$. Note that in the latter case

$$0 > M_n - C_n \geq S_n - C_n = -|S_n - C_n| \geq - \max_{1 \leq k \leq n} |S_k - C_k|. \quad (1)$$

Otherwise, if $M_n \geq C_n$, then since $C_n \uparrow$ we have $C_k \leq C_n$ for all $k \leq n$. Thus for $k \leq n$

$$S_k - C_k \leq S_k - C_k \leq |S_k - C_k|,$$

hence

$$\max_{1 \leq k \leq n} (S_k - C_n) \leq \max_{1 \leq k \leq n} |S_k - C_k|,$$

or equivalently

$$0 \leq M_n - C_n \leq \max_{1 \leq k \leq n} |S_k - C_k|. \quad (2)$$

Combining (1) and (2) we obtain

$$|M_n - C_n| \leq \max_{1 \leq k \leq n} |S_k - C_k|.$$

Therefore for all $N < n$

$$\frac{|M_n - C_n|}{b_n} \leq \frac{\max_{1 \leq k \leq n} |S_k - C_k|}{b_n} \leq \frac{\max_{1 \leq k \leq N} |S_k - C_k|}{b_n} + \frac{\max_{N < k \leq n} |S_k - C_k|}{b_n} \leq \frac{\max_{1 \leq k \leq N} |S_k - C_n|}{b_n} + \frac{\max_{N < k \leq n} |S_k - C_k|}{b_n} \quad (\text{since } b_n \uparrow).$$

First letting $n \rightarrow \infty$, then $N \rightarrow \infty$ the desired result is obtained.
Theorem 2. If $C_n \leq 0$, $0 < b_n \uparrow \infty$, and $\frac{S_n - C_n}{b_n} \to 0$ a.s., then $\frac{M_n}{b_n} \to 0$ a.s.

Proof. For all $k \geq 1$

$$S_k \leq S_k - C_k \leq |S_k - C_k|,$$

whence $M_n \leq \max_{1 \leq k \leq n} |S_k - C_k|$.

Thus for all $N < n$

$$\frac{M_n}{b_n} \leq \frac{\max_{1 \leq k \leq n} |S_k - C_k|}{b_n} \leq \frac{\max_{1 \leq k \leq N} |S_k - C_k|}{b_n} + \max_{k > N} \frac{|S_k - C_k|}{b_k}.$$

Again, letting $n \to \infty$, then $N \to \infty$ we conclude that

$$\limsup_{n \to \infty} \frac{M_n}{b_n} \leq 0 \quad \text{a.s.} \quad (3)$$

On the other hand, since $M_n \geq Y_1$, we see that

$$\liminf_{n \to \infty} \frac{M_n}{b_n} \geq \liminf_{n \to \infty} \frac{Y_1}{b_n} = 0 \quad \text{a.s.} \quad (4)$$

Combining (3) and (4) we conclude that $\frac{M_n}{b_n} \to 0$ a.s.

3. Applications.

We can use these theorems in more general settings than just the usual i.i.d. case. In the first corollary note that neither independence, nor common distribution is assumed.

Corollary 1. Let $(X_n, \ n \geq 1)$ and $X$ be random variables such that $(X_n, \ n \geq 1)$ is stochastically dominated by $X$ in the sense that there exists a finite constant $D$ such that

$$P(|X_n| > t) \leq D P(|X| > t), \ t \geq 0, \ n \geq 1.$$
Let \((a_n, n \geq 1)\) and \((b_n, n \geq 1)\) be constants satisfying \(0 < b_n \uparrow \infty\) and
\[
\max_{1 \leq k \leq n} \frac{b_k}{|a_k|} \sum_{j=n}^{\infty} \frac{|a_j|}{b_j} = O(n).
\]
If \(\sum_{n=1}^{\infty} P(|a_n X| > b_n) < \infty\), then
\[
\max_{1 \leq k \leq n} \frac{\sum_{j=1}^{k} a_j X_j}{b_n} \rightarrow 0 \quad \text{a.s.}
\]

**Proof.** By Theorem 2 of [2] we see that \(S_n/b_n \rightarrow 0\) a.s., where \(S_n = \sum_{j=1}^{n} a_j X_j, \ n \geq 1\).
Thus by either Theorem 1 or 2 the desired result is obtained.

It's important to note that these results also hold for strong laws in Banach spaces.

**Corollary 2.** Let \((V_n, n \geq 1)\) be independent mean zero random elements in a real separable Rademacher type \(p(1 < p \leq 2)\) Banach space. Suppose that \((V_n, n \geq 1)\) is stochastically dominated by a random element \(V\) in the sense that
\[
P(\|V_n\| > t) \leq D P(\|DV\| > t), \quad t \geq 0, \ n \geq 1,
\]
for some finite constant \(D\), where \(E\|V\|^q < \infty\) for some \(1 \leq q < p\). If \((a_n, n \geq 1)\) and \((b_n, n \geq 1)\) are constants satisfying \(0 < b_n \uparrow \infty\), \(\sum_{k=1}^{\infty} |a_k| = O(b_n)\), and \(a_n/b_n = O(n^{-1/2})\), then
\[
\max_{1 \leq k \leq n} \left\| \sum_{j=1}^{k} a_j V_j \right\|_{b_n} \rightarrow 0 \quad \text{a.s.}
\]

**Proof.** By Theorem 6 of [4] we have \(\|S_n\|/b_n \rightarrow 0\) a.s., where \(S_n = \sum_{j=1}^{n} a_j V_j, n \geq 1\). Thus
\[
\frac{\max_{1 \leq k \leq n} \|S_k\|}{b_n} \leq \frac{\max_{1 \leq k \leq n} \|S_k\|}{b_n} + \max_{k>n} \frac{\|S_k\|}{b_k} \rightarrow 0 \quad \text{a.s.}
\]
as \(n \rightarrow \infty\), then \(N \rightarrow \infty\).

For an interesting application we present
Example 1. Let \((X_n, n \geq 1)\) be i.i.d. random variables with density
\[ f(x) = cx^{-2}(\log x)^{-1}I_{(\epsilon, \infty)}(x), \quad -\infty < x < \infty, \] where \(c\) is a suitably chosen constant. Then
\[ \max_{1 \leq k \leq n} \frac{\sum_{j=1}^{k} \frac{1}{j \log j} X_j}{(\log_2 n)^2} \to \frac{c}{2} \text{ a.s.} \]

Proof. Let \(S_n = \sum_{j=1}^{n} \frac{1}{j \log j} X_j\) and \(b_n = (\log_2 n)^2\), \(n \geq 1\). Then, by Example 3.2 of [1], \(S_n/b_n \to c/2\) a.s. Thus \((S_n - C_n)/b_n \to 0\) a.s., where \(C_n = \frac{c}{2}(\log_2 n)^2\), \(n \geq 1\). Therefore, via Theorem 1, we see that \((M_n - C_n)/b_n \to 0\) a.s.

We conclude with an example that in a way "corrects" a strange phenomenon. For, it was shown in [3] that \(S_n/b_n \to \) nonzero constant a.s., where \(S_n\) is the sum of independent mean zero random variables.

Example 2. Let \((X_n, n \geq 1)\) be i.i.d. random variables with common density \(f(x) = cx^{-2}(\log x)^{-3}I_{(\epsilon, \infty)}(x), \quad -\infty < x < \infty,\) where \(c\) is a suitably chosen constant. Then
\[ \max_{1 \leq k \leq n} \frac{\sum_{j=1}^{k} \frac{(\log j)^2}{j} \left( X_j - \frac{c}{2} \right)}{\log n} \to 0 \text{ a.s.} \]

Proof. Let \(S_n = \sum_{j=1}^{n} \frac{(\log j)^2}{j} \left( X_j - \frac{c}{2} \right)\) and \(b_n = \log n\), \(n \geq 1\). Thus by Example 1 of [3], \(S_n/b_n \to -c/2\) a.s. Hence \((S_n - C_n)/b_n \to 0\) a.s., where \(C_n = \frac{-c}{2}(\log n), n \geq 1\). This combined with Theorem 2 implies that \(M_n/b_n \to 0\) a.s.

References.


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