FAITHFUL HOMOGENEOUS SPACES OVER COMMUTATIVE 
GROUPS AND TST-SPACES

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ABSTRACT

A set $Q$ is a faithful homogeneous space over a commutative group iff there 
is a family $S$ of mappings such that $(Q,S)$ is a TST-space.

If a group $(G, \circ)$ of mappings acts strictly transitively on a given set $X$, then $X$ is 
said to be a faithful homogeneous space over the group $(G, \circ)$, in accordance with [1].

In [2] the notion of a parallelogram space on a given set is defined and it is proved 
that there is the structure of a parallelogram space on a given set $Q$ iff $Q$ is a faithful 
homogeneous space over a commutative group.

In [3] the notion of a TST-space is defined (see [4]). A TST-space $(Q,S)$ is a nonvoid 
set $Q$ with a family $S$ of its involutory mappings (which are called symmetries) such that 
$S$ acts transitively on $Q$ and from $\sigma_1, \sigma_2, \sigma_3 \in S$ it follows $\sigma_3 \circ \sigma_2 \circ \sigma_1 \in S$, where $\circ$ is the 
composition of mappings. In the same paper [3] it is proved that there is the structure of 
a parallelogram space on a given set $Q$ iff there is a family $S$ of its mappings such that 
$(Q,S)$ is a TST-space.

From these two results it follows immediately the following theorem.

Theorem. A set $Q$ is a faithful homogeneous space over a commutative group $(T, \circ)$ iff 
there is a family $S$ of its mappings such that $(Q,S)$ is a TST-space.

But, we shall give here a direct proof of this theorem.
Let $Q$ be a faithful homogeneous space over a commutative group $(T, \circ)$, i.e. let this group act strictly transitively on the set $Q$.

For any $a, b \in Q$ let us denote by $\tau_{a,b}$ the element of $T$ such that $\tau_{a,b}(a) = b$. Then we have identities

$$\tau^{-1}_{a,b} = \tau_{b,a},$$

$$\tau_{a,b} \circ \tau_{b,c} = \tau_{a,c} \circ \tau_{a,b} = \tau_{a,c}$$

and for any $a \in Q$ the mapping $\tau_{a,a}$ is the identity. Now, for any $a, b \in Q$ we define a symmetry $\sigma_{a,b} : Q \to Q$ by the equivalence

$$(1) \quad \sigma_{a,b}(c) = d \iff \tau_{a,c}(d) = b,$$

which can be written in the form

$$(2) \quad \sigma_{a,b}(c) = d \iff \tau_{a,c} = \tau_{d,b}.$$ 

From $\sigma_{a,b}(c) = d$, i.e. $\tau_{a,c} = \tau_{d,b}$, it follows by the commutativity of the group $(T, \circ)$ successively

$$(\tau_{d,b} \circ \tau_{c,b})(a) = (\tau_{a,c} \circ \tau_{c,b})(a) = (\tau_{c,b} \circ \tau_{a,c})(a) = \tau_{c,b}(c) = b,$$

i.e.

$$\tau_{c,b}(a) = \tau_{a,b}^{-1}(b) = d,$$

or $\tau_{c,b} = \tau_{a,d}$. But, $\tau_{a,d} = \tau_{c,b}$ implies by (2) $\sigma_{a,b}(d) = c$. Therefore, from $\sigma_{a,b}(c) = d$ it follows $\sigma_{a,b}(d) = c$ and every symmetry is an involutory mapping. Because $\tau_{a,a} = \tau_{b,b}$ (the identity), from (2) we conclude that $\sigma_{a,b}(a) = b$ and the family $S$ of all symmetries acts transitively on the set $Q$. We shall show that this action is strict. Indeed, let $a, b \in Q$ be any elements. It suffices to show that $\sigma_{c,b}(a) = b$ implies $\sigma_{c,d} = \sigma_{a,b}$, i.e. that from
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$\sigma_{c,d}(x) = y$ it follows $\sigma_{a,b}(x) = y$. According to (1) we must prove that from $\tau_{a,b}(b) = d$ and $\tau_{c,d}(y) = d$ it follows $\tau_{a,c}(y) = b$. But, we obtain successively

$$\tau_{a,c}(y) = (\tau_{a,c} \circ \tau_{c,d})(y) = \tau_{a,c}(d) = \tau_{c,d}^{-1}(d) = b.$$ 

Therefore, for any $a, b \in Q$ and any $\sigma \in S$ from $\sigma(a) = b$ it follows $\sigma = \sigma_{a,b}$. Now, let $\sigma_1, \sigma_2, \sigma_3 \in S$ be any symmetries. Let $a \in Q$ be any element and $b = \sigma_1(a), c = \sigma_2(b), d = \sigma_3(c)$. Then $\sigma_1 = \sigma_{a,b}, \sigma_2 = \sigma_{b,c}, \sigma_3 = \sigma_{c,d}$. We shall prove the equalities

$$\sigma_{b,c} \circ \sigma_{a,b} = \tau_{a,c}, \quad \sigma_{d,c} \circ \sigma_{a,d} = \tau_{a,c}. \quad (3)$$

Let $x \in Q$ be any element and $y = \sigma_{a,b}(x), z = \sigma_{b,c}(y)$. Then we have by (2) $\tau_{a,c} = \tau_{y,b}, \quad \tau_{b,y} = \tau_{x,c}$ and it follows

$$\tau_{a,c} = \tau_{a,c} \circ \tau_{x,c} = \tau_{y,b} \circ \tau_{x,c} = \tau_{y,b} \circ \tau_{x,c} = \tau_{y,b} \circ \tau_{x,c} = \tau_{y,b} \circ \tau_{x,c} = \tau_{y,b} \circ \tau_{x,c} = \tau_{x,z},$$

i.e.

$$\tau_{a,c}(x) = z = \sigma_{b,c}(y) = (\sigma_{b,c} \circ \sigma_{a,b})(x).$$

Therefore, we have the first equality (3). The second equality (3) can be proved analogously. But, from (3) it follows

$$\sigma_{a,d} = \sigma_{d,c}^{-1} \circ \sigma_{b,c} \circ \sigma_{a,b} = \sigma_{c,d} \circ \sigma_{b,c} \circ \sigma_{a,b} = \sigma_3 \circ \sigma_2 \circ \sigma_1,$$

i.e. $\sigma_3 \circ \sigma_2 \circ \sigma_1$ is a symmetry. We have proved the following proposition.

**Proposition 1.** If $Q$ is a faithful homogeneous space over a commutative group $(T, \circ)$ and if $S$ is the set of all symmetries defined by (1), then $(Q, S)$ is a TST-space.

Now, let $(Q, S)$ be any TST-space, i.e. let $S$ be a family of involutory mappings (symmetries) of the set $Q$ which acts transitively on $Q$ and such that $\sigma_1, \sigma_2, \sigma_3 \in S$ implies $\sigma_3 \circ \sigma_2 \circ \sigma_1 \in S$. 
The involutivity of symmetries implies obviously their bijectivity. Let $T$ be the family of all compositions of two symmetries. If $\sigma_2 \circ \sigma_1, \sigma_4 \circ \sigma_3 \in T$, then

$$(\sigma_4 \circ \sigma_3) \circ (\sigma_2 \circ \sigma_1)^{-1} = \sigma_4 \circ \sigma_3 \circ \sigma_1 \circ \sigma_2 \in T.$$  

because of $\sigma_3 \circ \sigma_1 \circ \sigma_2 \in S$. Therefore, $(T, \circ)$ is a subgroup of the group generated by all symmetries. Moreover, we have

$$(\sigma_4 \circ \sigma_3) \circ (\sigma_2 \circ \sigma_1) = \sigma_4 \circ (\sigma_3 \circ \sigma_2 \circ \sigma_1)^{-1} = \sigma_4 \circ \sigma_1 \circ \sigma_2 \circ \sigma_3$$

$$= (\sigma_4 \circ \sigma_1 \circ \sigma_2)^{-1} \circ \sigma_3 = (\sigma_2 \circ \sigma_1) \circ (\sigma_4 \circ \sigma_3),$$

and $(T, \circ)$ is a commutative group. In [3] (see [4]) it is proved that the family $S$ acts strictly transitively on $Q$, i.e. that for any $\sigma_1, \sigma_2 \in S$ and any $a \in Q$ from $\sigma_1(a) = \sigma_2(a)$ it follows $\sigma_1 = \sigma_2$. Then transitivity of $T$ on $Q$ follows immediately from the transitivity of $S$ on $Q$. But, in [3] it is proved that $T$ acts strictly transitively on $Q$. Indeed, let $\tau_1(a) = \tau_2(a)$ for any $\tau_1, \tau_2 \in T$ and any $a \in Q$. Then, for some $\sigma \in S$ we have $(\sigma \circ \tau_1)(a) = (\sigma \circ \tau_2)(a)$. But, $\sigma \circ \tau_1, \sigma \circ \tau_2 \in S$ and hence $\sigma \circ \tau_1 = \sigma \circ \tau_2$, i.e. $\tau_1 = \tau_2$. Therefore, $Q$ is a faithful homogeneous space over the group $(T, \circ)$. For any $a, b \in Q$ let $\sigma_{a,b} \in S$ resp. $\tau_{a,b} \in T$ be such elements that $\sigma_{a,b}(a) = a$ resp. $\tau_{a,b}(a) = b$. Then for any $a, b, c \in Q$ we obtain

$$(\sigma_{b,c} \circ \sigma_{a,b})(a) = \sigma_{b,c}(b) = c = \tau_{a,c}(a),$$

i.e. $\sigma_{b,c} \circ \sigma_{a,b} = \tau_{a,c}$. Let us prove the equivalence (1). Let $\sigma_{a,b}(c) = d$. Then we have $\sigma_{a,b}(d) = c$ and from $\sigma_{b,c}(b) = c$ it follows $\sigma_{b,c}(c) = b$. Therefore, we get

$$\tau_{a,c}(d) = (\sigma_{b,c} \circ \sigma_{a,b})(d) = \sigma_{b,c}(c) = b.$$  

Thus, we have proved the following proposition.

**Proposition 2.** If $(Q, S)$ is a TST-space and $T$ the family of all compositions of two elements of $S$, then $(T, \circ)$ is a commutative group, $Q$ is a faithful homogeneous space over it and the equivalence (1) holds.
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Propositions 1 and 2 prove our Theorem.

References.


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