A NOTE ON THE p-DISTRIBUTIVITY
IN NON-ARCHIMEDEAN f-RINGS

Joan Trias Pairó

ABSTRACT
Non-archimedean f-rings need not be p-distributive. Moreover, if \( \{d_i\}_i \) is a subset of a non-archimedean f-ring and \( a \geq 0 \), the elements \( \forall d_i \) and \( \forall a d_i \) need not be equal. We prove, however, that the difference is an infinitely small element when the ring has a strong unity.

Recall first from [5] that a lattice-ordered ring \( A \) is left p-distributive (respectively right p-distributive) if whenever \( \forall d_i \) exists (with \( \{d_i\}_i \subseteq A \)) and \( a \geq 0 \), then \( \forall a d_i \) (resp. \( \forall d_i a \)) also exists and \( \forall d_i = \forall a d_i \) (resp. \( (\forall d_i a) = \forall (d_i a) \)). A is p-distributive if it is left and right p-distributive.

Archimedean f-rings are p-distributive [3]. With independence of the hypothesis of archimedeanity it is possible, however, to find p-distributive f-rings: a) Commutative f-rings with unity, in which every non-unit is a zero-divisor, are p-distributive [5]. b) Bounded inversion f-rings are also p-distributive, as is shown immediately from propositions 1 and 2 and lemma 1.
of [5]. We recall that an f-ring is of bounded inversion if every $x \geq 1$ is a unit.

It may be asked whether there exist non p-distributive non-archimedean f-rings or not. Some examples related with the question follow:

**Example 1.** Let $X$ be a non-pseudocompact topological space and let $C(X)$ be the f-ring of real continuous functions defined on $X$, under pointwise ordering and operations. Let $M$ be a hyper-real maximal ideal [2] and $A = C(X)/M$ the canonically ordered quotient ring. Then $A$ is a totally ordered non-archimedean field. Since every $x > 0$ is a unit, $A$ is p-distributive [5].

**Example 2.** Let $\mathbb{R}[x]$ be the ring of polynomials in an indeterminate $x$ with real coefficients, endowed with the usual operations and the total ordering defined as follows: if $P(x) = a_nx^n + \ldots + a_0$ ($a_n \neq 0$), then $P > 0$ if and only if $a_n > 0$. We thus obtain a non-archimedean f-ring that is not p-distributive, since for example $x(\sum_{n=1}^{\infty} n^{-1}) = 0$ and the infimum of $\{x_n^{-1} \mid n \in \mathbb{N}\}$ does not exist.

In the non p-distributive case, a natural question arises: which is the relation, if there is any, between $\forall i \in I$ and $\forall a_i$, if we suppose that both suprema exist?

Before giving an answer to this question, recall that an element $x \in A$ is infinitely small with respect to $y \in A$ whenever $n|x| \leq |y|$ holds for every $n \in \mathbb{N}$ [4] (x<<y, for short). If $I_0(y) = \{x \in A \mid x < y\}$, we write $I_0(A) = \bigcup_{y \in A} I_0(y)$. Also, an element is said to be a strong unity if it is contained in no proper solid subgroup [3].

**Definition.** The elements $x, y$ of a lattice-ordered ring $A$ are called infinitely close if $x - y \in I_0(A)$.

We now state the main result of this note:

**Theorem.** Let $A$ be a non archimedean lattice-ordered ring with a strong unity $u$. Then:
A note on the p-distributivity in non-archimedean f-rings

a) \(I_0(A)\) is a closed solid ideal.

b) If \(A\) is besides an f-ring, \(a \geq 0\), and \(\{d_i\}_i\) is a subset of \(A\) such that \(\forall d_i, \forall a d_i \) (respectively \(\forall d, a \)) exist, then \(\exists d_i \) and \(\exists a d_i \) are infinitely close (respectively, so are \(\forall d_i \) and \(\forall a d_i \)).

Proof. Note first that \(I_0(A) \subseteq I_0(u)\). Indeed, if \(z \ll g\) for some \(g \geq 0\), there exists \(n \in \mathbb{N}\) such that \(n^{-1}|z| \ll n^{-1}|u|\) holds for every \(n \in \mathbb{N}\). So \(n|z| \ll |u|\) for every \(n \in \mathbb{N}\). Now,

a) It is clear that \(I_0(A)\) is a solid ideal (in the ring-theoretic sense); let now \(\{x_j\}_j\) be a subset of \(I_0(A)\) such that \(x = \bigvee_j x_j\) exists in \(A\). We must prove that \(x \in I_0(A)\). By the preceding remark, we have \(n x_j \ll |u|\) and \(n x_j \ll |u|\), \(\forall n \in \mathbb{N}, \forall j\). Hence, using \(x^+ = \bigvee_j x_j^+\) and \(x^- = \bigwedge_j x_j^-\), we obtain \(x \in I_0(A)\).

b) Since \(I_0(A)\) is closed (a)), the canonical mapping of \(A\) onto \(A/I_0(A)\) preserves the suprema of subsets of \(A\) \(\{4\}\). By the remark above, \(I_0(A/I_0(A)) = 0\), and so \(A/I_0(A)\) is an archimedean f-ring. Hence it is p-distributive, and this completes the proof.

References


Dept. de Matemàtiques i Estadística
E.T.S. d'Arquitectura
Diagonal, 649,
Barcelona-28, Spain.