BOLSHEV’S METHOD OF CONFIDENCE LIMIT CONSTRUCTION

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Confidence intervals and regions for the parameters of a distribution are constructed, following the method due to L.N. Bolshev. This construction method is illustrated with Poisson, exponential, Bernouilli, geometric, normal and other distributions depending on parameters.

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1. REGIONS, INTERVALS, CONFIDENCE LIMITS

Let \( X = (X_1, \ldots, X_n)^T \) be a sample with realizations \( x = (x_1, \ldots, x_n)^T, x \in X \subseteq R^n \). Suppose that \( X_i \) has a density \( f(x; \theta), \quad \theta = (\theta_1, \ldots, \theta_k)^T \in \Theta \subseteq R^k \), with respect to the Lebesgue measure,

\[
H_0: X_i \sim f(x; \theta), \quad \theta = (\theta_1, \ldots, \theta_k)^T \in \Theta \subseteq R^k.
\]

Let \( b = b(\theta) \) be a function \( b(\cdot): \Theta \Rightarrow B \subseteq R^m \), \( B^0 \) is the interior of \( B \).

**Definition 1.** A random set \( C(\mathbb{X}) \), \( C(\mathbb{X}) \subseteq B \subseteq R^m \) is called the confidence region for \( b = b(\theta) \) with the confidence level \( \gamma \) \((0.5 < \gamma < 1)\) if

\[\inf_{\theta \in \Theta} P_{\theta}\{C(\mathbb{X}) \ni b(\theta)\} = \gamma.\]

This definition implies for all \( \theta \in \Theta \)

\[P_{\theta}\{C(\mathbb{X}) \ni b(\theta)\} \geq \gamma.\]

In the case \( b(\theta) \in B \subseteq R^1 \) the confidence region is often an interval in \( R^1 \),

\[C(\mathbb{X}) = [b_1(\mathbb{X}), b_2(\mathbb{X})] \subseteq B \subseteq R^1,\]

and it is called the confidence interval with the confidence level \( \gamma \) for \( b \). The statistics \( b_1(\mathbb{X}) \) and \( b_2(\mathbb{X}) \) are called the confidence limits of the confidence interval \( C(\mathbb{X}) \).

**Definition 2.** A statistic \( b_1(\mathbb{X}) \) \((b_2(\mathbb{X}))\) is called the inferior (superior) confidence limit with the confidence level \( \gamma_1 (\gamma_2) \) \((\text{or inferior (superior) } \gamma_1 (\gamma_2) - \text{ confidence limit briefly})\), if

\[\inf_{\theta \in \Theta} P_{\theta}\{b_1(\mathbb{X}) < b\} = \gamma_1 \left( \inf_{\theta \in \Theta} P_{\theta}\{b_2(\mathbb{X}) > b\} = \gamma_2 \right), \quad 0.5 < \gamma_j < 1\]

The \( \gamma = 1 - \alpha \) confidence interval has the form \([b_1(\mathbb{X}), b_2(\mathbb{X})]\), where \( b_1(\mathbb{X}) \) and \( b_2(\mathbb{X}) \) are the \( \gamma_1 = 1 - \alpha_1 \) inferior and \( \gamma_2 = 1 - \alpha_2 \) superior confidence limits, respectively, such that \( \alpha_1 + \alpha_2 = \alpha \), \((0 < \alpha < 0.5)\). If \( \alpha_1 = \alpha_2 \), then take \( \gamma_1 = \gamma_2 = 1 - \alpha/2 \).

**Definition 3.** The intervals

\[\{b_1(\mathbb{X}), +\infty\} \quad \text{and} \quad \{-\infty, b_2(\mathbb{X})\}\]

are called the superior and inferior confidence intervals for \( b \). Both intervals are unilateral.
2. THEOREM OF BOLSHEV

Lemma (Bolshev) Let $G(t)$ be the distribution function of the random variable $T$. Then for all $z \in [0, 1]$

\[ P\{G(T) \leq z\} \leq z \leq P\{G(T - 0) < z\}. \]

If $T$ is continuous, then

\[ P\{G(T) = z\} = z, \quad 0 \leq z \leq 1. \]

Proof: First, we prove the inequality

\[ P\{G(T) \leq z\} \leq z, \quad 0 \leq z \leq 1. \]

If $z = 1$, then $P\{G(T) \leq 1\} \leq 1$. Fix $z \in [0, 1)$ and for this value of $z$ consider the different cases.

1) There exists a solution $y$ of the equation $G(y) = z$. Note

\[ y_0 = \sup\{y : G(y) = z\}. \]

It can be:

a) $G(y_0) = z$. In this case

\[ P\{G(T) \leq z\} \leq P\{T \leq y_0\} = G(y_0) = z. \]

b) $G(y_0) > z$. Then

\[ P\{G(T) \leq z\} \leq P\{T < y_0\} = G(y_0 - 0) \leq z. \]

2) A solution of the equation $G(y) = z$ does not exist. In this case there exists $y$ such that

\[ G(y) > z \quad \text{et} \quad G(y - 0) < z, \]

so

\[ P\{G(T) \leq z\} \leq P\{T < y\} = G(y - 0) < z. \]

The inequality (2) is proved.
We prove now the second inequality in (1):

\[(3) \quad z \leq P\{G(T - 0) < z\}, \quad 0 \leq z \leq 1.\]

Consider the statistic \(-T\). Its distribution function is

\[G^- (y) = P\{-T \leq y\} = P\{T \geq -y\} = 1 - G(-y - 0).\]

Replacing \(T, z, G\) by \(-T, 1 - z\) and \(G^-\) in the inequality (2) we have:

\[P\{G^-(-T) \leq 1 - z\} \leq 1 - z, \quad 0 \leq z \leq 1.\]

This implies

\[
P\{1 - G(T - 0) \leq 1 - z\} \leq 1 - z, \\
P\{G(T - 0) \geq z\} \leq 1 - z, \\
P\{G(T - 0) < z\} \geq z, \quad 0 \leq z \leq 1.
\]

If \(T\) is continuous, then \(G(t - 0) = G(t)\), and (2) and (3) imply \(P\{G(T) \leq z\} = z\) for all \(z \in [0, 1]\).

The lemma is proved.

\[\blacksquare\]

**Theorem** (Bolshev) Suppose that the random variable \(T = T(\mathbb{R}, b), b \in B\), is such that its distribution function

\[G(t; b) = P_b\{T \leq t\}\]

depends only on \(b\) for all \(t \in \mathbb{R}\) and the functions

\[I(b; x) = G(T|x, b) - 0; b) \quad \text{and} \quad S(b; x) = G(T(x, b); b)\]

are decreasing and continuous in \(b\) for all fixed \(x \in X\). In this case:

1) the statistic \(b_1(\mathbb{R})\) such that

\[(4) \quad b_1 = b_1(\mathbb{R}) = \sup\{b : I(b; \mathbb{R}) \geq \gamma; b \in B\}, \quad \text{if this supremum exists},\]
or

\[(5) \quad b_i = b_i(X) = \inf B, \text{ otherwise} \]

is the inferior confidence limit for \(b \in B^0\) with confidence level larger or equal to \(\gamma\);

2) the statistic \(b_s(X)\) such that

\[(6) \quad b_s = b_s(X) = \inf \{b : S(b;X) \leq 1 - \gamma, \quad b \in B\}, \quad \text{if this infimum exists}, \]

or

\[(7) \quad b_s = b_s(X) = \sup B, \text{ otherwise} \]

is the superior confidence limit for \(b \in B^0\) with the confidence level larger or equal to \(\gamma\).

3) if \(x \in X\), is such that the functions \(I(b;x)\) and \(S(b;x)\) are strongly decreasing with respect to \(b\), then \(b_i(x)\) and \(b_s(x)\) are the roots of the equations

\[(8) \quad I(b_i(x);x) = \gamma \quad \text{and} \quad S(b_s(x);x) = 1 - \gamma\]

**Proof:** Denote \(D = D(X)\) the event

\[D = \{\text{there exists } b \text{ such that } I(b;X) \geq \gamma\}.\]

Then for the true value \(b \in B^0\) we have (using Bolshev’s lemma)

\[
P\{b_i < b\} = P\{(b_i < b) \cap D\} + P\{|b_i < b| \cap \bar{D}\} =
\]

\[
P\{(\sup b^* : I(b^*;X) \geq \gamma, b^* \in B) < b \cap D\} + P\{\inf B < b \cap \bar{D}\} =
\]

\[
P\{I(b;X) < \gamma \cap D\} + P\{\bar{D}\} \geq P\{I(b;X) < \gamma \cap D\} + P\{I(b;X) < \gamma \cap \bar{D}\} =
\]

\[
P\{I(b;X) < \gamma\} \geq \gamma.
\]

The theorem is proved. \(\blacksquare\)

**Remark:** Often, instead of the statistic \(T\) a sufficient statistic or some function of a sufficient statistic for a parameter \(b\) can be taken. \(\square\)

553
3. EXAMPLES

1. Let $X = (X_1, \ldots, X_n)^T$ be a sample and suppose that $X_i$ has a Poisson distribution with a parameter $\theta$:

$$X_i \sim f(x; \theta) = \frac{\theta^x}{x!} e^{-\theta}, \quad x \in X = \{0, 1, \ldots \}, \quad \Theta = [0, \infty[.$$ 

Denote

$$T = X_1 + \ldots + X_n.$$ 

a) Show that the statistics

$$\theta_i = \frac{1}{2n} \chi^2_{1-\gamma_i}(2T) \quad \text{and} \quad \theta_s = \frac{1}{2n} \chi^2_{s}(2T + 2)$$

are the inferior and superior confidence limits for $\theta$ with confidence levels larger or equal to $\gamma_1$ and $\gamma_2$ respectively; $\chi^2_{\alpha}(n)$ denotes the $\alpha$-quantile of a chi-square distribution with $n$ degrees of freedom.

b) Find a confidence interval for $\theta$ with confidence level larger or equal to $\gamma$.

**Solution.** The sufficient statistic $T$ follows the Poisson distribution with parameter $n\theta$. Then

$$G(k; \theta) = P_\theta \{T \leq k\} = \sum_{i=0}^{k} \frac{(n\theta)^i}{i!} e^{-n\theta} = P\{\chi^2_{k+1} \geq 2n\theta\} = \mathcal{P}(2n\theta, 2k+2), \quad k = 0, 1, \ldots$$

and

$$G(k - 0; \theta) = P_\theta \{T < k\} = \sum_{i=0}^{k-1} \frac{(n\theta)^i}{i!} e^{-n\theta} = \mathcal{P}(2n\theta, 2k), \quad k = 1, 2, \ldots,$$

$$G(k - 0; \theta) = 0, \quad k = 0.$$ 

The functions $I$ and $S$ are

$$I(\theta; \mathbb{X}) = \begin{cases} \mathcal{P}(2n\theta, 2T), & \text{if } \mathbb{X} \neq 0, \\ 0, & \text{if } \mathbb{X} = 0, \end{cases}$$

$$S(\theta; \mathbb{X}) = \mathcal{P}(2n\theta, 2T + 2).$$
The function $S$ is strictly decreasing for all $T, T > 0$, and $I$ is strictly decreasing for all $T \neq 0$. In these cases the theorem of Bolshev implies (see (8)):

$$P(2n\theta_1, 2T) = \gamma_1 \quad P(2n\theta_2, 2T + 2) = 1 - \gamma_2,$$

from which it follows

(9) \quad $\theta_1 = \frac{1}{2n}x^2_{1-\gamma_1}(2T), \quad \theta_2 = \frac{1}{2n}x^2_{\gamma_2}(2T + 2)$.

If $T = 0$ then $I(\theta; X) = 0$. There is no such $\theta$ that

$$I(\theta; X) = \gamma_1 > \frac{1}{2}.$$

The formula (5) implies

$$\theta_i = \inf_{\theta > 0} \theta = \inf[0, +\infty] = 0.$$

b) The interval $[\theta_i, \theta_s]$ is the confidence interval for $\theta$ with a confidence level larger or equal to $\gamma = 1 - \alpha$, if $\gamma_1 = 1 - \alpha_1, \gamma_2 = 1 - \alpha_2, \alpha_1 + \alpha_2 = \alpha$. If $\alpha_1 = \alpha_2$, take $\gamma_1 = \gamma_2 = 1 - \alpha/2$.

2. Let $X = (X_1, \ldots, X_n)^T$ be a sample and suppose that $X_i$ has an exponential distribution with mean $\theta, \theta > 0$:

(10) \quad $X_i \sim f(x; \theta) = \frac{1}{\theta} \exp\left\{-\frac{x}{\theta}\right\} I_{(x > 0)}$.

a) Find $\gamma$-confidence limits for $\theta$.

b) Let $X_n^{(r)} = (X_{(1)}, \ldots, X_{(r)})^T$ be a type II censored sample from the distribution (10).

Find a $\gamma$-confidence interval for $\theta$ and the survival function

$$S(x; \theta) = P_{X_1 > x}.$$

\textbf{Solution.} a). Denote

$$T = X_1 + \ldots + X_n.$$

The sufficient statistic $T$ follows a gamma distribution $G(n; 1/\theta)$ with parameters $n$ and $1/\theta$:

$$P\{T \leq t\} = \frac{1}{(n-1)!\theta^n} \int_0^t u^{n-1}e^{-u/\theta}du, \quad t \geq 0.$$
and hence \( T/\theta \) follows the gamma distribution \( G(n;1) \), and
\[
\frac{2T}{\theta} = \chi^2_{2n}
\]
In this example the functions \( I \) and \( S \) can be taken as
\[
I(\theta;X) = S(\theta;X) = 1 - F_p(\frac{2T}{\theta}, 2n) \]
These functions are decreasing in \( \theta \) and the formula (8) implies
\[
1 - F_p(\frac{2T}{\theta_1}, 2n) = \gamma \quad \text{and} \quad 1 - F_p(\frac{2T}{\theta_2}, 2n) = 1 - \gamma,
\]
from where we obtain
\[
\frac{2T}{\theta_1} = \chi^2_{\gamma}(2n) \quad \text{and} \quad \frac{2T}{\theta_2} = \chi^2_{1-\gamma}(2n),
\]
and hence
\[
\theta_1 = \frac{2T}{\chi^2_{\gamma}(2n)} \quad \text{and} \quad \theta_2 = \frac{2T}{\chi^2_{1-\gamma}(2n)}
\]

b) As it is well known the statistic
\[
T_r = \sum_{k=1}^{r} X_{(k)} + (n-r)X_{(r)}
\]
follows a gamma distribution \( G(r; \frac{1}{\theta}) \), and hence the \( \gamma = 1 - \alpha \)-confidence interval for \( \theta \) is \( [\theta_1, \theta_2] \), where
\[
\theta_1 = \frac{2T_r}{\chi^2_{\gamma/2}(2r)} \quad \text{and} \quad \theta_2 = \frac{2T_r}{\chi^2_{1-\gamma/2}(2r)}
\]
Since the survival function \( S(x;\theta) = e^{-x/\theta}, x > 0 \), is increasing in \( \theta \), we have the \( \gamma \)-confidence interval \( [S_i, S_s] \) for \( S(x;\theta) \), where
\[
S_i = e^{-x/\theta_1} \quad \text{and} \quad S_s = e^{-x/\theta_2}.
\]

3. Let \( X = (X_1, \ldots, X_n)^T \) be a sample from Bernoulli distribution with parameter \( \theta \):
\[
X_i \sim f(x;\theta) = \theta^i(1-\theta)^{1-x}, x \in X = \{0,1\}, \theta \in \Theta = [0,1].
\]
Find the limits of confidence for \( \theta \) with the confidence levels larger or equal to \( \gamma \).
Solution. It is clear that the sufficient statistic

\[ T = \sum_{i=1}^{n} X_i \]

follows the binomial distribution \( B(n, \theta) \) with parameters \( n \) and \( \theta \). Then

\[ G(k; \theta) = P_{\theta}(T \leq k) = \sum_{i=0}^{k} \binom{n}{i} \theta^i (1-\theta)^{n-i} = \]

\[ I_{1-\theta}(n-k, k+1) = 1 - I_{\theta}(k+1, n-k), \quad k = 0, 1, \ldots, n-1, \]

\[ G(k; \theta) = 1, \text{ if } k = n, \]

where \( I_\alpha(a, b) \) is the beta distribution function with parameters \( a \) and \( b \), and

\[ G(k-0; \theta) = \sum_{i=0}^{k-1} \binom{n}{i} \theta^i (1-\theta)^{n-i} = \]

\[ 1 - I_{\theta}(k, n-k+1), \quad k = 1, 2, \ldots, n, \]

\[ G(k-0; \theta) = 0, \text{ if } k = 0. \]

The functions \( I \) and \( S \) are

\[ I(\theta; \mathcal{X}) = \begin{cases} I_{1-\theta}(n-T+1, T), & \text{if } T \neq 0 \\ 0, & \text{otherwise} \end{cases}, \]

\[ S(\theta; \mathcal{X}) = \begin{cases} I_{1-\theta}(n-T, T+1), & \text{if } T \neq n \\ 1, & \text{if } T = n. \end{cases} \]

We remark that \( S(\theta; \mathcal{X}) \) is strictly decreasing in \( \theta \) for \( T \neq n \), and \( I(\theta; \mathcal{X}) \) is strictly decreasing in \( \theta \) for \( T \neq 0 \), and hence from the formula (8) it follows that

\[ I_{1-\theta}(n-T+1, T) = \gamma_1 \quad \text{for } T \neq 0 \]

and

\[ \theta_i = 0, \text{ if } T = 0, \]

\[ I_{1-\theta}(n-T, T+1) = 1 - \gamma_1 \quad \text{for } T \neq n \]

and

\[ \theta_s = 1, \text{ if } T = n. \]

Hence,

\[ \theta_i = \begin{cases} 1 - x(\gamma_1; n-T+1, T), & \text{if } T \neq 0 \\ 0, & \text{if } T = 0, \end{cases} \]

\[ \theta_s = \begin{cases} 1 - x(1-\gamma_1; n-T, T+1), & \text{if } T \neq n \\ 1, & \text{if } T = n, \end{cases} \]

557
where \(x(y_1; a, b)\) is the \(y_1\)-quantile of the beta distribution with parameters \(a\) and \(b\).

4. Let \(X\) be a discrete random variable with the cumulative distribution function

\[
F(x; \theta) = P_\theta\{X \leq x\} = (1 - \theta^{|x|})1_{[0, +\infty]}(x), \quad x \in \mathbb{R}^1, \quad \theta \in \Theta = [0, 1].
\]

Find a \(\gamma\)-confidence interval for \(\theta\), if \(X = 1\).

**Solution.** In this case

\[
I(X; \theta) = F(X - 0; \theta) \quad \text{and} \quad S(X; \theta) = F(X; \theta).
\]

If \(X = 1\) then

\[
I(1; \theta) = F(1 - 0; \theta) = F(0; \theta) = 0
\]

and according to the formula (5) we have that the inferior confidence limit \(\theta_i\) for \(\theta\) with confidence level larger or equal to \(\gamma_1\) is

\[
\theta_i = \inf \theta = \inf[0, 1[ = 0.
\]

If \(\gamma_1 = 1\) then \(P\{\theta_i \leq \theta\} = \gamma_1\), so \(\theta_i = 0\) is 1-confidence inferior limit for \(\theta\). On the other hand the function

\[
S(1; \theta) = F(1; \theta) = 1 - \theta
\]

is decreasing in \(\theta\), and hence according to the formula (8) we have

\[
S(1; \theta_s) = 1 - \gamma_2,
\]

from where \(\theta_s = \gamma_2\), so the \(\gamma_1 = 1\) and \(\gamma_2\) confidence limits for \(\theta\) are 0 and \(\gamma_2\), and a *gamma*-confidence interval for \(\theta\) is \([0, \gamma]_\gamma\), since for \(\gamma_1 = 1\) the equality \(\gamma = \gamma_1 + \gamma_2 - 1\) is true when \(\gamma_2 = \gamma\).

5. Let \(X_1\) and \(X_2\) be two independent random variables,

\[
X_i \sim f(x; \theta) = e^{-(x-\theta)}1_{(0, +\infty]}(x), \quad \theta \in \Theta = \mathbb{R}^1.
\]

Find the smallest \(\gamma\)-confidence interval for \(\theta\).

**Solution.** The likelihood function \(L(\theta)\) for \(X_1\) and \(X_2\) is

\[
L(\theta) = \exp\{-(X_1 + X_2 - 2\theta)\}1_{[0, +\infty]}(X_1),
\]

558
from where it follows that \(X(1) = \min(X_1, X_2)\) is the minimal sufficient statistic for \(\theta\) and \(\hat{\theta} = X(1)\) is the maximum of the function

\[
l(\theta) = \ln L(\theta) = (2\theta - X_1 - X_2)1_{[\theta,\infty)}(X(1)),
\]

which is increasing in \(\theta\) on the interval \([-\infty, X(1)]\). Since for any \(x \geq 0\)

\[
P_0\{X(1) > x\} = P_0\{X_1 > x, X_2 > x\} = \left(\int_x^\infty e^{-(x-\theta)} \, dt\right)^2 = e^{-2(x-\theta)},
\]

we have

\[
P_0\{X(1) \leq x\} = G(x; \theta) = \left(1 - e^{-2(x-\theta)}\right)1_{[\theta,\infty)}(x), \quad x \in \mathbb{R}^1.
\]

In this example the functions \(I(\theta; X(1))\) and \(S(\theta; X(1))\) are

\[
I(\theta; X(1)) = S(\theta; X(1)) = G(X(1); \theta) = 1 - e^{-2(X(1) - \theta)}.
\]

They are decreasing in \(\theta\) and hence from the theorem of Bolshev we have

\[
1 - e^{-2(X(1) - \theta)} = \gamma_1, \quad \text{and} \quad 1 - e^{-2(X(1) - \theta_i)} = 1 - \gamma_2,
\]

thus

\[
\theta_i = X(1) + \frac{1}{2} \ln (1 - \gamma_1), \quad \text{and} \quad \theta_s = X(1) + \frac{1}{2} \ln \gamma_2.
\]

The interval \([\theta_i, \theta_s]\) is the \(\gamma\)-confidence interval for \(\theta\) if \(\gamma = \gamma_1 + \gamma_2 - 1\).

The length of this interval is

\[
\theta_s - \theta_i = \frac{1}{2} \left[ \ln \gamma_2 - \ln (1 - \gamma_1) \right].
\]

We have to find \(\gamma_1\) and \(\gamma_2\) such that \(\gamma_1 + \gamma_2 = 1 + \gamma\), \(0 < \gamma_i \leq 1\) \((i = 1, 2)\) and the interval \([\theta_i, \theta_s]\) is the shortest. We consider \(\theta_s - \theta_i\) as the function of \(\gamma_2\). In this case

\[
(\theta_s - \theta_i)' = \frac{1}{2} \left[ \ln \gamma_2 - \ln \gamma_2 - \gamma' \right] = \frac{1}{2} \left( \frac{1}{\gamma_2} - \frac{1}{\gamma_2 - \gamma} \right) < 0,
\]

and hence \(\theta_s - \theta_i\) is decreasing in \(\gamma_2\) \((0.5 < \gamma_2 \leq 1)\) and the minimal value of \(\theta_s - \theta_i\) occurs when \(\gamma_2 = 1\) and \(\gamma_1 = 1 + \gamma - \gamma_2 = \gamma\). Since in this case

\[
\theta_i = X(1) + \frac{1}{2} \ln (1 - \gamma) \quad \text{and} \quad \theta_s = X(1)
\]

559
\[
\min (\theta, \theta_0) = -\frac{1}{2} \ln (1 - \gamma) - \ln \sqrt{1 - \gamma}
\]

6. Let \(X_1\) and \(X_2\) be two independent random variables uniformly distributed on \([\theta - 1, \theta + 1]\), \(\theta \in R^1\). Find the shortest \(\gamma\)-confidence interval for \(\theta\).

**Solution.** It is clear that \(Y_i - \theta\) is uniformly distributed on \([-1, 1]\), from where it follows that the distribution of the random variable

\[
T = X_1 + X_2 - 2\theta = Y_1 + Y_2
\]

does not depend on \(\theta\). It is easy to show that

\[
G(y) = P\{T \leq y\} = \begin{cases} 
0, & y \leq -2, \\
\frac{1}{y(y+2)^2}, & -2 \leq y \leq 0, \\
1 - \frac{(y-2)^2}{8}, & 0 \leq y \leq 2, \\
1, & y \geq 2.
\end{cases}
\]

The function

\[
G(T) = G(X_1 + X_2 - 2\theta), \theta \in R^1,
\]

is decreasing in \(\theta\). From (8) it follows that the inferior and the superior confidence limits with the confidence levels \(\gamma_1\) and \(\gamma_2\) correspondingly (\(0.5 < \gamma_i < 1\)) satisfy the equations

\[
G(X_1 + X_2 - 2\theta_1) = \gamma_1 \quad \text{and} \quad G(X_1 + X_2 - 2\theta_2) = 1 - \gamma_2,
\]

from where we find

\[
\theta_1 = \frac{X_1 + X_2}{2} - 1 + \sqrt{2(1 - \gamma_1)} \quad \text{and} \quad \theta_2 = \frac{X_1 + X_2}{2} + 1 - \sqrt{2(1 - \gamma_2)}.
\]

It is easy to show that for given \(\gamma = \gamma_1 + \gamma_2 - 1\) the function

\[
\theta_1 - \theta_2 = 2 - \sqrt{2(1 - \gamma_1)} - \sqrt{2(1 - \gamma_2)}
\]

has its minimal value (considered as function of \(\gamma_1\), \(0.5 < \gamma_1 < 1\)) when

\[
\gamma_1 = \frac{1 + \gamma}{2}
\]

In this case \(\gamma_2 = \frac{1 - \gamma}{2}\), so the shortest \(\gamma\)-confidence interval for \(\theta\) is \(\theta_1, \theta_2\) where

\[
\theta_1 = \frac{X_1 + X_2}{2} - 1 + \sqrt{1 - \gamma} \quad \text{and} \quad \theta_2 = \frac{X_1 + X_2}{2} + 1 - \sqrt{1 - \gamma}
\]

7. Suppose that \(T\) is the number of shots until the first success. Find the \(\gamma = 0.9\) confidence intervals for the probability \(p\) of success, if
a). \(T = 1\); b). \(T = 4\); c). \(T = 10\).
Solution. The distribution of $T$ is geometric:

$$P\{T = k\} = p(1 - p)^{k-1}, \ k = 1, 2, \ldots.$$

The values of the distribution function of $T$ in the points $k$ are

$$G(k; p) = \sum_{i=1}^{k} p(1 - p)^{i-1} = 1 - (1 - p)^{k-1}, \ k = 1, 2, \ldots.$$

The functions $I$ and $S$ are

$$I(p; T) = 1 - (1 - p)^{T-1}, \quad S(p; T) = 1 - (1 - p)^{T}.$$

The functions $I(p; T)$ and $S(p; T)$ are increasing in $p$ if $T > 1$ and $T < 1$, respectively. So they are decreasing in $q = 1 - p$.

It follows from the formula (8) that lower and upper confidence limits satisfy the equations

$$1 - q_i^{T-1} = \gamma_i \quad \text{for} \quad T > 1,$$

$$1 - q_s^T = 1 - \gamma_i \quad \text{for} \quad T < 1.$$

So

$$q_i = (1 - \gamma_i)^{\frac{1}{T-1}} \quad \text{for} \quad T > 1, \quad q_s = \gamma_i^{\frac{1}{T}} \quad \text{for} \quad T < 1.$$

and

$$p_i = 1 - q_i = 1 - \gamma_i^{\frac{1}{T}} \quad \text{for} \quad T > 1, \quad p_s = 1 - q_s = 1 - (1 - \gamma_i)^{\frac{1}{T}} \quad \text{for} \quad T > 1.$$

If $T = 1$, then $q_i = \inf \{0, 1\} = 0$, $p_s = 1$.

To find the $\gamma = 1 - \alpha = 0.9$ confidence interval we take $\gamma_i = 1 - \alpha/2 = 1 + \gamma/2 = 0.95$.

So the $\gamma = 0.9$ confidence interval for $p$ is $(p_i, p_s)$, where

$$p_i = 0.05, \quad p_s = 1 \quad \text{for} \quad T = 1,$$

$$p_i = 1 - 0.95^{\frac{1}{T}} = 0.01274, \quad p_s = 1 - 0.051^{\frac{1}{3}} = 0.6316 \quad \text{for} \quad T = 4,$$

$$p_i = 1 - 0.9^{\frac{1}{T}} = 0.005116, \quad p_s = 1 - 0.051^{\frac{1}{9}} = 0.2831 \quad \text{for} \quad T = 10.$$

8. Let $X = (X_1, \ldots, X_n)^T$ be a sample and suppose that $X_i$ has the normal distribution: $X_i \sim N(\mu, \sigma^2)$. Find a $\gamma$ confidence interval for $\mu$.

Solution. The sufficient statistic is $(\bar{X}, S^2)$

$$\bar{X} = \frac{1}{n} \sum_{i=1}^{n} X_i, \quad S^2 = \frac{1}{n-1} \sum_{i=1}^{n} (X_i - \bar{X})^2.$$
Consider the statistic
\[
T(\bar{X}, \mu) = \frac{\sqrt{n}(\bar{X} - \mu)}{S}.
\]

The random variable \( T(\bar{X}, \mu) \) has the Student distribution with \( n - 1 \) degrees of freedom and distribution function \( F_{n-1} \). So
\[
I(\mu, \bar{X}) = S(\mu, \bar{X}) = F_{n-1}(T(\bar{X}, \mu)).
\]

The functions \( I \) and \( S \) are decreasing with respect to \( \mu \), so by the theorem of Bolshev
\[
F_{n-1}(T(\bar{X}, \mu)) = \gamma_1 = \frac{1 + \gamma}{2},
\]
\[
F_{n-1}(T(\bar{X}, \mu_*)) = 1 - \gamma_1 = \frac{1 - \gamma}{2},
\]
and
\[
\mu_i = \bar{X} - \frac{S}{\sqrt{n}} t_{n-1} \left( \frac{1 + \gamma}{2} \right),
\]
\[
\mu_* = \bar{X} + \frac{S}{\sqrt{n}} t_{n-1} \left( \frac{1 + \gamma}{2} \right),
\]
where \( t_{n-1}(\alpha) \) is the \( \alpha \)-quantile of the Student distribution with \( n - 1 \) degrees of freedom.

Confidence intervals for the variance, for the difference of two means, for the ratio of two variances, etc., can be obtained in a similar way.

REFERENCES


PROBLEMES PROPOSATS

PROBLEMA N° 66

Sean \((X,Y)\) dos variables aleatorias discretas que toman los valores \((x_i,y_j)\), \(i = 1,\ldots,p, \ j = 1,\ldots,q\), con probabilidad no nula. Sea \(P\) la matriz \(p \times q\) con las densidades de probabilidad de \(Y\) condicionadas a \(X\)

\[
P = \begin{pmatrix}
    f(y_1/x_1) & \cdots & f(y_q/x_1) \\
    \vdots & \ddots & \vdots \\
    f(y_1/x_p) & \cdots & f(y_q/x_p)
  \end{pmatrix}
\]

Sea análogamente \(Q = (f(x_i/y_j))\), \(i = 1,\ldots,p, \ j = 1,\ldots,q\), la matriz \(q \times p\) con las densidades de \(X\) condicionadas a \(Y\).

Supongamos que \(a_i(X), \ i = 1,\ldots,p, \ b_j(Y), \ j = 1,\ldots,q\), son asignaciones de los valores de las variables tales que

\[
a = (a_1(X),\ldots,a_p(X))^\prime \\
b = (b_1(X),\ldots,b_q(X))^\prime
\]

verifican

\[
a = \beta Pb \quad b = \beta Qa
\]

siendo \(\beta > 0\). Se pide:

1. Probar que \(a\) es un vector propio de \(PQ\) de valor propio \(\lambda = 1/\beta^2\).
2. Probar que \(\lambda = 1\) es un valor propio de \(PQ\) de vector propio \(1_p = (1,\ldots,1)^\prime\).
3. Probar que \(0 \leq \lambda \leq 1\).

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PROBLEMA N° 67

Sea $X$ una variable aleatoria con función de distribución absolutamente continua $F_{\theta}(X)$ y función de densidad $f_{\theta}(x)$, donde $\theta$ es un parámetro desconocido. Supongamos que $f_{\theta}(x)$ cumple las condiciones de regularidad (el soporte de $X$ no depende del parámetro $\theta$, es posible derivar bajo el signo integral, etc.)

Sea

$$I(\theta) = E \left\{ \left( \frac{\partial}{\partial \theta} \log f_{\theta}(X) \right)^2 \right\}$$

la cantidad de información de Fisher.

Probar que se verifica la desigualdad

$$4 \left( \frac{\partial}{\partial \theta} F_{\theta}(x) \right)^2 \leq I(\theta)$$

uniformemente en $x$.

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