Compact Hypersurfaces with Constant Higher Order Mean Curvatures

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A fundamental question about hypersurfaces in the Euclidean space is to decide if the sphere is the only compact hypersurface (embedded or immersed) with constant higher order mean curvature $H_r$, for some $r = 1, \ldots, n$.

If the hypersurface $M^n$ is star-shaped, Hsiung [3] solved affirmatively the problem for any $r$. In particular if the Gauss-Kronecker curvature $H_n$ is constant, then $M^n$ is a sphere, because in this case the Hadamard theorem asserts that $M^n$ is convex. The convex case was studied previously by Liebmann [5] and Süss [9]. If the mean curvature $H_1$ is constant and $M^n$ is embedded, Aleksandrov [1] proved that $M^n$ is a sphere. In the immersed case Hsiang, Teng and Yu [4] and Wente [10] constructed non-spherical examples in higher dimension and in $\mathbb{R}^3$ respectively. If the scalar curvature $H_2$ is constant and the hypersurface is embedded we proved in [8] that it must be a sphere. In this paper we extend this result to higher order mean curvatures. In particular we prove that

«The sphere is the only embedded compact hypersurface in the Euclidean space with $H_r$ constant for some $r = 1, \ldots, n$.»

In this paper we use as in [8] a method of Reilly [7]. Recently with S. Montiel [6] we obtained a different proof of the above theorem. Another proof has been published by N. Korevaar [11].
1. Preliminaries

Let $\psi: M^n \to \mathbb{R}^{n+1}$ be an orientable hypersurface immersed in the Euclidean space. Let $N$ be an unit normal vector field on $M$, $\sigma$ the second fundamental form of $M$ with respect to $N$ and $k_i$, $i = 1, \ldots, n$ the principal curvatures of $M$.

For any $r = 1, \ldots, n$ we define the mean curvature of order $r$, $H_r$, by the identity

$$ (1 + tk_1) \cdots (1 + tk_n) = 1 + \binom{n}{1} H_1 t + \binom{n}{2} H_2 t^2 + \cdots + \binom{n}{n} H_n t^n $$

for any real number $t$. For instance, $H_1$ is simply the mean curvature $H_1 = H = (k_1 + \cdots + k_n)/n$. $H_2$ is, up a constant, the scalar curvature and $H_n = k_1 k_2 \cdots k_n$ is the Gauss-Kronecker curvature. From the Gauss equation we have that if $r$ is even, then $H_r$ is an intrinsic invariant of $M$. Note that for the unit sphere, and with respect to the unit inner normal, we have $H_r = 1$ for any $r$. By convenience we put $H_0 = 1$. These curvatures satisfy a basic relation in global hypersurface theory, which is stated in the following lemma.

**Lemma** (Minkowski Formulæ [3]). Let $\psi: M^n \to \mathbb{R}^{n+1}$ be a compact orientable hypersurface immersed in the Euclidean space. Then for any $r = 1, \ldots, n$ we have

$$ \int_M H_{r-1} \, dA + \int_M H_r \langle \psi, N \rangle \, dA = 0. $$

Let $\Omega^{n+1}$ be a compact Riemannian manifold with smooth boundary $M^n = \partial \Omega$. Let $dV$ and $dA$ be the canonical measures on $\Omega$ and $M$ respectively and $V$ and $A$ the volume of $\Omega$ and the area of $M$. Given $f$ in $C^2(\Omega)$ we denote $z = f|_M$ and $u = \partial f/\partial N$, where $N$ is the inner unit normal on $M$. Reilly’s formula [7] states that

$$ \int_\Omega ([\tilde{\Delta} f]^2 - |\tilde{\nabla} f|^2 - \text{Ric} (\tilde{\nabla} f, \tilde{\nabla} f)) \, dV = \int_M [-2(\Delta z) u + nHu^2 + \sigma(\nabla z, \nabla z)] \, dA, $$

where $\tilde{\nabla} f$ and $\tilde{\Delta} f$ being the gradient, the Laplacian and the Hessian of $f$ in $\Omega$, $\text{Ric}$ the Ricci curvature of $\Omega$, $\nabla z$ and $\Delta z$ the gradient and the Laplacian of $z$ in $M$, and $\sigma$ and $H$ the second fundamental form and the mean curvature of $M$ with respect to $N$.

If $M$ is a compact hypersurface embedded in $\mathbb{R}^{n+1}$, then $M$ is the boundary of a compact domain $\Omega \subset \mathbb{R}^{n+1}$. So if $x$ denotes the position vector in $\mathbb{R}^{n+1}$, then $\Delta |x|^2 = 2(n+1)$, and from the divergence theorem we have

$$ (n+1) V + \int_M \langle \psi, N \rangle \, dA = 0, $$

where $\psi$ being the immersion of $M$ in $\mathbb{R}^{n+1}$. 

2. An inequality

For the next result we will follow closely the ideas of Reilly [7].

**Theorem 1.** Let $\Omega^{n+1}$ be a compact Riemannian manifold with smooth boundary $M$ and non-negative Ricci curvature. Let $H$ be the mean curvature of $M$. If $H$ is positive everywhere, then

\[ \int_M \frac{1}{H} \, dA \geq (n + 1) V. \]

The equality holds if and only if $\Omega$ is isometric to an Euclidean ball.

**Proof.** Let $f$ in $C^\infty(\bar{\Omega})$ be the solution of the Dirichlet problem

\[
\begin{cases}
\Delta f = 1 \text{ in } \Omega, \\
\tau = 0 \text{ on } M.
\end{cases}
\]

From the divergence theorem we have

\[ V = \int_\Omega \Delta f \, dV = -\int_M u \, dA. \]

Combining Schwarz inequality $(\Delta f)^2 \leq (n + 1) |\nabla f|^2$ with (3) we obtain

\[ \frac{V}{n + 1} \geq \int_M Hu^2 \, dA. \]

Finally, from (6), Schwarz inequality and (7) it follows that

\[
V^2 = \left( \int_M u \, dA \right)^2 = \left( \int_M (uH^{-1/2})H^{-1/2} \, dA \right)^2 \\
\leq \int_M u^2 H \, dA \int_M H^{-1} \, dA \\
\leq \frac{V}{n + 1} \int_M \frac{1}{H} \, dA,
\]

and we have proved inequality (5).

If equality holds, then $\nabla^2 f$ is proportional to the metric everywhere. As $\Delta f = 1$, we conclude that

\[ \nabla^2 f = \frac{1}{n + 1} \langle \cdot, \cdot \rangle. \]
Deriving covariantly we obtain $\bar{\nabla}^3 f = 0$ and from the Ricci-identity,

\[(9) \quad R(X, Y)\bar{\nabla} f = 0,\]

for any $X$, $Y$ tangent vector to $\Omega$, where $R$ is the curvature of $\Omega$. From the maximum principle $f$ attains its minimum at some point $x_0$ in the interior of $\Omega$. From (8) it follows that

$$\bar{\nabla} f = \frac{1}{n + 1} r \frac{\partial}{\partial r},$$

where $r$ is the distance to the point $x_0$, which combined with (9), Cartan’s Theorem and the fact that $f$ vanishes at the boundary of $\Omega$ implies that $\Omega$ is an Euclidean ball whose center is $x_0$, and $f$ is given by

$$f(x) = [2(n + 1)]^{-1} (|x - x_0|^2 - a^2)$$

in $\Omega$, $a$ being the radius of the ball.

3. Hypersurfaces with $H_r$ Constant

In this section we prove the main result of this paper. Given $k = 1, \ldots, n$ we consider the function $\sigma_k : \mathbb{R}^n \to \mathbb{R}$ defined by

$$\sigma_k(x_1, \ldots, x_n) = \text{elementary symmetric polynomial of degree } k \text{ in the variables } x_1, \ldots, x_n.$$

Thus $H_k = \sigma_k(k_1, \ldots, k_n)$. We denote by $C_k$ the connected component of the set $\{x \in \mathbb{R}^n / \sigma_k(x) > 0\}$ which contains the vector $(1, \ldots, 1)$. From Gårding [2] we have that if $k < r$, then $C_k \supset C_r$. Moreover if $x \in C_k$ we have for $k \leq r$ that $\sigma_k(x)^{(k-1)/k} \leq \sigma_{k-1}(x)$. For $k \geq 2$ equality holds if and only if $x$ is proportional to $(1, \ldots, 1)$. Clearly if $x_i > 0$ for any $i$, then $(x_1, \ldots, x_n) \in C_k$.

**Theorem 2.** Let $M^n$ be a compact hypersurface embedded in the Euclidean space $\mathbb{R}^{n+1}$. If $H_r$ is constant for some $r = 1, \ldots, n$, then $M^n$ is a sphere.

**Proof.** As $M$ has an elliptic point, $H_r$ must be a positive constant. As the principal curvatures are continuous functions we have that $(k_1, \ldots, k_n) \in C_r$ at any point. Hence $(k_1, \ldots, k_n) \in C_k$ for $k$ smaller than $r$. In particular $H_k > 0$ for $k < r$. Moreover

\[(10) \quad H_k^{(k-1)/k} \leq H_{k-1} \quad k = 1, \ldots, r,\]

As consequence

\[(11) \quad H_r^{1/r} \leq H \text{ in } M.\]
Now we use Minkowski formulae and (3):
\[
0 = \int_M H_{r-1} \, dA + \int_M H_r \langle \psi, N \rangle \, dA \\
= \int_M H_{r-1} \, dA + H_r \int_M \langle \psi, N \rangle \, dA \\
= \int_M H_{r-1} \, dA - (n+1)H_r V.
\]
Combining this equality with (10) we have
\[
(n+1)H_r V = \int_M H_{r-1} \, dA \geq AH_{r}^{\frac{r}{r-1}},
\]
and so
\[
H_{r}^{\frac{1}{r}} \leq \frac{A}{(n+1)V}.
\]
On the other hand from (5) and (11) we obtain
\[
(n+1)V \leq \int_M \frac{dA}{H} \leq AH_{r}^{-\frac{1}{r}},
\]
which is
\[
H_{r}^{\frac{1}{r}} \leq \frac{A}{(n+1)V},
\]
and the equality holds if and only if $M$ is a sphere. The theorem follows from (12) and (13).

4. An Extension of the Aleksandrov Theorem

First we observe that a compact hypersurface embedded in the Euclidean space is a critical point of the isoperimetric functional if and only if it has constant mean curvature.

Theorem 3. Let $\bar{M}^{n+1}$ be a Riemannian manifold with non-negative Ricci curvature, and let $\Omega$ be a compact domain in $\bar{M}$ with smooth boundary. If $\Omega$ is a critical point of the isoperimetric functional
\[
\Omega \to \frac{A(\partial \Omega)^{n+1}}{V(\Omega)^{\frac{n+1}{n}}},
\]
then $\Omega$ is isometric to an Euclidean ball.
PROOF. Given a smooth function $f$ on $\partial \Omega$, we consider the normal variation of $\partial \Omega$ defined by $\psi_t : \partial \Omega \to \tilde{M}$, $\psi_t(p) = \text{Exp}_p(-tf(p)N(p))$, where $\text{Exp}$ is the exponential map of $\tilde{M}$. $\psi_t$ determine a variation of $\Omega$, $\Omega_t$ for $|t| < \epsilon$. We put $V(t) = V(\Omega_t)$ and $A(t) = A(\partial \Omega_t)$. The first variation formulae of the functionals above are given by

$$A'(0) = n \int_{\partial \Omega} fH dA,$$

$$V'(0) = \int_{\partial \Omega} f dA.$$

By hypothesis we have

$$\frac{d}{dt} \bigg|_{t=0} \frac{A(t)^{n+1}}{V(t)^n} = 0,$$

or equivalently

$$\int_{\partial \Omega} f((n+1) VH - A) dA = 0, \text{ for any } f.$$

Then $H = A/(n+1)V$ and we have equality in (5). Therefore $\Omega$ is isometric to an Euclidean ball.

Remark. Let $\psi : M^n \to \mathbb{R}^{n+1}$ be an immersed compact hypersurface. Suppose that $M$ is the boundary of a certain manifold $\Omega^{n+1}$, and that the immersion $\psi$ extends to an immersion of $\Omega$, $\tilde{\psi} : \Omega^{n+1} \to \mathbb{R}^{n+1}$. It is easy to see that Aleksandrov proof [1] can be adapted to this situation: If $M^n$ has constant mean curvature, it must be a sphere.

U. Pinkall pointed out to me that Reilly's method can also be used in this case. In fact, taking on $\Omega$ the pull-back of the Euclidean metric, inequality (5) remains true and the same holds for identity (3). On the other hand, Minkowski formulae hold for any immersed hypersurface. So theorem 2 extends to the above type of immersed hypersurfaces.

References


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