Eigenvalue Problems of Quasilinear Elliptic Systems on $\mathbb{R}^n$

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Abstract

In this paper, we get the existence results of the nontrivial weak solution $(\lambda, u)$ of the following eigenvalue problem of quasilinear elliptic systems

$$-D_a(a_{ab}(x, u)D_b u) + \frac{1}{2} D_a a_{ab}(x, u)D_a u' D_b u' + h(x)u' = \lambda|u|^{p-2}u',$$

for $x \in \mathbb{R}^n$, $1 \leq i \leq N$ and

$$u = (u^1, u^2, \ldots, u^N) \in E = \{ v = (v^1, v^2, \ldots, v^N) \mid v^i \in H^1(\mathbb{R}^n), 1 \leq i \leq N \},$$

where $a_{ab}(x, u)$ satisfy the natural growth conditions. It seems that this kind of problem has never been dealt with before.

1. Introduction

We consider eigenvalue problems of the following quasilinear elliptic systems on $\mathbb{R}^n$

$$(1.1) \quad -D_a(a_{ab}(x, u)D_b u) + \frac{1}{2} D_a a_{ab}(x, u)D_a u' D_b u' + h(x)u' = \lambda|u|^{p-2}u',$$
for \( x \in \mathbb{R}^n \), \( 1 \leq i \leq N \) and

\[
u = (u^1, u^2, \ldots, u^N) \in E = \{ v = (v^1, v^2, \ldots, v^N) \mid v^i \in H^1(\mathbb{R}^N), 1 \leq i \leq N \}
\]

where \( R < p < 2\hat{n}/(\hat{n} - 2) \), \( \hat{n} = n \) if \( n > 2 \), \( 2\hat{n}/(\hat{n} - 2) \) is any positive number larger than 2 if \( n \leq 2 \),

\[
D_{\alpha} = \frac{\partial}{\partial x_{\alpha}}, \quad D_{\mu} = \frac{\partial}{\partial u^{\mu}} \quad (1 \leq \alpha \leq n, \quad 1 \leq i \leq N)
\]

and the summation conventions have been used and will be used in the following, i.e. the repeated Greek letters and Latin letters denote the sum from 1 to \( n \) and 1 to \( N \) respectively.

Problem (1.1) comes from the theory of harmonic mappings. There have been some results of (1.1) in bounded domains ([1], [2]). In [1], the existence of solutions for (1.1) is discussed under the conditions

\[
\mu_1 |\xi|^2 \leq a_{\alpha\beta}(x, u) \xi_\alpha \xi_\beta \leq \mu_2 |\xi|^2 \quad \mu_1, \mu_2 > 0
\]

\[
\lim_{u \to +\infty} uD_\alpha a_{\alpha\beta}(x, u) = 0
\]

for every \( (u, \xi) \in \mathbb{R}^1 \times \mathbb{R}^n \), \( x \in \Omega \subset \mathbb{R}^n \), where \( N = 1 \), \( p = 2n/(n - 2) \), \( n > 2 \) if \( n > 2 \). In [2] the existence theorem is obtained when \( N \geq 1 \), \( h = 0 \), \( 2 < p \leq 2n/2n - 2 \), \( n > 2 \) under the conditions

\[
\begin{align*}
\left[ a_1 |\xi|^2 \leq \sigma(|u|)|\xi|^2 \leq a_{\alpha\beta}(x, u) \xi_\alpha \xi_\beta \leq a_2 \sigma(|u|)|\xi|^2 \\
|D_\alpha uD_{\mu} a_{\alpha\beta}(x, u)| \leq C\sigma(|u|) \\
|D_{\mu} a_{\alpha\beta}(x, u)| \leq C\sigma(|u|), \quad |D_\alpha a_{\alpha\beta}(x, u)| \leq \eta(|u|) \\
-\frac{1}{2} D_{\mu} a_{\alpha\beta}(x, u) \xi_\alpha \xi_\beta \leq a_3 a_{\alpha\beta}(x, u) \xi_\alpha \xi_\beta \quad (0 < a_3 < 1),
\end{align*}
\]

for every \( (x, u, \xi) \in \Omega \times \mathbb{R}^N \times \mathbb{R}^n \), where \( \sigma(t) \), \( \eta(t) \) are nonnegative continuous functions on \([0, +\infty)\) satisfying that for any \( c_1 > 1 \), there exists \( c_2 \), such that \( \sigma(c_1 t) \leq c_2 \sigma(t) \) for all \( t \geq 0 \).

However, there have not been any results for (1.1) in the unbounded domain \( \mathbb{R}^n \). Formally, if the minimum of the functional

\[
I(u) = \int_{\mathbb{R}^n} [a_{\alpha\beta}(x, u)D_\alpha uD_{\beta} u^i + h(x)|u^i|^2] \, dx
\]

over the set \( \{ u \in E \mid \int_{\mathbb{R}^n} |u|^p \, dx = \mu \} \) (\( \mu > 0 \)) were achieved by some \( u \), there should be a \( \lambda \in \mathbb{R}^1 \) such that \( (\lambda, u) \) solves (1.1) in a weak sense. But there are some difficulties in dealing with the functional \( I(u) \). Firstly, because of the unboundedness of \( \mathbb{R}^n \), the Sobolev embedding is not compact and the standard convex-compactness techniques can not be used, at least in a straightforward
way as in the case of bounded domains, and this makes the problem of the existence of a minimizer more difficult. Secondly, the space where \( I \) is differentiable is \( L^\alpha \cap E \) (see [3]), so even if we had found a minimizer \( u \in I \) of \( I \), we could not conclude the existence of \( (\lambda, u) \in \mathbb{R}^1 \times E \) solving (1.1), unless we had known that \( u \in L^\alpha \). But, usually, the fact that \( |u|_\infty \) is finite is obtained because \( u \) satisfies the related Euler equation which in turn is a consequence of the differentiability of \( I \) at \( u \). This makes the problem complicated.

To overcome the first difficulty, we use the concentration compactness principle, recently developed by P. L. Lions ([4], [5]), when treating the constrained variational problems in unbounded domains. To overcome the second difficulty, we first show that, for any minimizer \( u \) of \( I \) and some \( \varphi \in E \),

\[
\frac{d}{dt} I(u + t\varphi) \bigg|_{t=0} = 0
\]

i.e. the Euler equation related to the functional \( I \) holds in a weak sense for \( u \) over special test functions in \( E \). We then use the Nash-Moser methods to show that \( |u|_\infty \) is finite and finally we get the existence of a nontrivial solution \( (\lambda, u) \) of (1.1).

2. Main Results

In this section, we present the main results of this paper. First of all, we give some notations and conditions.

Let \( H^1(\mathbb{R}^n) \) be the usual Sobolev space, \( N \geq 1 \) be a natural number and \( E = \{ u = (u^1, u^2, \ldots, u^N) | u^i \in H^1(\mathbb{R}^n), 1 \leq i \leq N \} \). The scalar product of \( u, v \in E \) is defined by

\[
\langle u, v \rangle = \int_{\mathbb{R}^n} [D_\alpha u D_\alpha v + u \cdot v] \, dx
\]

and \( (E, \langle \cdot, \cdot \rangle) \) is a Hilbert space, the norm of \( u \in E \) is \( |u|_E = (\|Du\|^2 + |u|^2)^{1/2} \) where hereafter \( \|f\|_E \) denotes the \( L^2(\mathbb{R}^n) \) norm of the function \( f \) and \( |f| \) denotes the Euclidean norm of the function \( f \) (possibly vector valued). For simplicity, we denote \( |u|_E \) by \( |u| \) for \( u \in E \).

The main conditions imposed on (1.1) will be the following

(i) \( 2 < p < 2\hat{n}/(\hat{n} - 2) \) where \( \hat{n} = n \) if \( n > 2 \); and \( 2\hat{n}/(\hat{n} - 2) \) is any positive number larger than \( 2 \) if \( n \leq 2 \).

(ii) \( a_{\alpha\beta}(x, u) \in C^1(\mathbb{R}^n \times \mathbb{R}^N), \ a_{\alpha\beta} = a_{\beta\alpha} \) for any \( \alpha, \beta \) and \( a_1 > 0, a_2 > 1 \) such that for any \( (x, u, \xi) \in \mathbb{R}^n \times \mathbb{R}^N \times \mathbb{R}^n \)

\[
a_1 |\xi|^2 \leq \sigma(|u|)|\xi|^2 \leq a_{\alpha\beta}(x, u)\xi_\alpha \xi_\beta \leq a_2 \sigma(|u|)|\xi|^2
\]

(2.1)
holds, where \( \sigma(t) \) is a nonnegative nondecreasing continuous function on 
\([0, +\infty)\) satisfying: for any \( l > 1 \), there exists \( C_l > 0 \), such that

\[
\sigma(lt) \leq C_l \sigma(t), \quad \text{for all } t \geq 0
\]

and \( C_l \) are bounded whenever \( l \) are bounded. Moreover, there is a constant \( C > 0 \) with

\[
\sigma(t) \leq C(1 + |t|^{q})
\]

where \( 0 \leq q \leq 4/(n - 2) \) if \( n > 2 \) and \( 0 \leq q \) if \( n \leq 2 \).

(iii) \( a_{\alpha\beta}(x, u) \rightarrow \bar{a}_{\alpha\beta}(u) \) as \( |x| \rightarrow +\infty \) uniformly for \( u \) bounded.

(iv) There exists, \( s \geq 0, s < p - 2 \) such that

\[
a_{\alpha\beta}(x, \lambda u)\xi_\alpha \xi_\beta \leq \lambda^{s} a_{\alpha\beta}(x, u)\xi_\alpha \xi_\beta
\]

\[
a_{\alpha\beta}(x, u)\xi_\alpha \xi_\beta \leq \bar{a}_{\alpha\beta}(u)\xi_\alpha \xi_\beta
\]

for any \((x, \xi, u) \in \mathbb{R}^{n} \times \mathbb{R}^{N} \times \mathbb{R}^{n}\), where \( p \) is given in (i) and \( \bar{a}_{\alpha\beta} \) are defined in (iii), and \( \lambda > 1 \) is arbitrary.

(v) \( h \in C(\mathbb{R}^{n}) \) and there are \( \bar{h}, c > 0 \) such that \( h(x) \geq c, h(x) \leq \bar{h} \) for any \( x \in \mathbb{R}^{n} \) and \( \lim_{|x| \to \infty} h(x) = \bar{h} \).

(vi) There is a constant \( c > 0 \) such that

\[
|u'D_{\alpha}a_{\beta\alpha}(x, u)| \leq c\sigma(|u|)
\]

\[
|D_{\alpha}a_{\beta\alpha}(x, u)| \leq c\eta(|u|)
\]

for any \((x, u) \in \mathbb{R}^{n} \times \mathbb{R}^{N}\), where \( \eta(t) \) is a nonnegative nondecreasing continuous function on \([0, +\infty)\) and \( \sigma(t) \) is given in (ii).

(vii) There is a constant \( a_3 \) with \( 0 < a_3 < 1 \) such that

\[
-\frac{1}{2} u'D_{\alpha}a_{\beta\alpha}(x, u)\xi_\alpha \xi_\beta \leq a_3 a_{\alpha\beta}(x, u)\xi_\alpha \xi_\beta
\]

for any \((x, u, \xi) \in \mathbb{R}^{n} \times \mathbb{R}^{N} \times \mathbb{R}^{n}\).

Remark 2.1. If \( a_{\alpha\beta}(x, u), h(x) \) satisfy (i)-(vii), then \( \bar{a}_{\alpha\beta}(u), \bar{h} \) satisfy (i)-(v).

If \( a_{\alpha\beta}(x, u), h(x) \) satisfy (i)-(v), we set, for any \( u \in E \)

\[
I(u) = \int_{\mathbb{R}^{n}} (a_{\alpha\beta}(x, u)D_{\alpha}u'D_{\beta}u + h(x)|u|^{2}) \, dx
\]

\[
I^{\infty}(u) = \int_{\mathbb{R}^{n}} (\bar{a}_{\alpha\beta}(u)D_{\alpha}u'D_{\beta}u + \bar{h}|u|^{2}) \, dx
\]

For any \( \lambda > 0 \), we set

\[
I_{\lambda} = \inf \left\{ I(u) \mid u \in E, \int_{\mathbb{R}^{n}} |u|^{p} \, dx = \lambda \right\}
\]

\[
I^{\infty}_{\lambda} = \inf \left\{ I^{\infty}(u) \mid u \in E, \int_{\mathbb{R}^{n}} |u|^{p} \, dx = \lambda \right\}
\]
It is clear that

\begin{align}
I_\lambda &= \inf \left\{ I(\lambda^{1/p} u) \mid u \in E, \int_{\mathbb{R}^n} |u|^p \, dx = 1 \right\} \\
I_\lambda^* &= \inf \left\{ I^*(\lambda^{1/p} u) \mid u \in E, \int_{\mathbb{R}^n} |u|^p \, dx = 1 \right\}
\end{align}

The pair \((\lambda, u) \in \mathbb{R}^1 \times E\) will be called a weak solution of (1.1) if

\[
\int_{\mathbb{R}^n} \left[ a_{\alpha\beta}(x, u) D_\alpha u^i D_\beta \varphi^i + \varphi^i D_\alpha (a_{\alpha\beta}(x, u) D_\beta u^i) + h(x) u^i \varphi^i \right] \, dx = \lambda \int_{\mathbb{R}^n} |u|^{p-2} u^i \varphi^i \, dx
\]

for any \(\varphi \in L^\infty \cap E\).

It is evident that \(u = 0\) is a trivial solution of (1.1) for any \(\lambda\).

The main results of this paper are the following

**Theorem 2.1.** Suppose that (i)-(vi) hold, then for any \(\lambda > 0\), \(I_\lambda^*\) is achieved by some \(u \in E\).

**Theorem 2.2.** Suppose that (i)-(vi) hold, then there is a \(\lambda_0 > 0\) such that \(I_{\lambda_0}\) is achieved by some \(u \in E\). Moreover, if \(I_\lambda < I_\lambda^*\) for any \(\lambda > 0\), then \(I_\lambda\) is achieved by some \(u \in E\) for any \(\lambda > 0\).

**Theorem 2.3.** Suppose that (i)-(vii) hold, then (1.1) possesses at least a nontrivial weak solution \((\lambda, u) \in \mathbb{R}^1 \times E\) and \(\|u\|_\infty < \infty\).

**Remark 2.2.** By (iv)-(v), it is trivial that \(I_\lambda \leq I_\lambda^*\), and by Theorem 2.1, \(I_\lambda < I_\lambda^*\) (for all \(\lambda > 0\)) if

\[\int_{\mathbb{R}^n} \left[ a_{\alpha\beta}(x, u) D_\alpha u^i D_\beta u^i + h(x) |u|^2 \right] \, dx < \int_{\mathbb{R}^n} \left[ \bar{a}_{\alpha\beta}(u) D_\alpha u^i D_\beta u^i + \bar{h} |u|^2 \right] \, dx\]

holds for \(u \in E, \int_{\mathbb{R}^n} |u|^p \, dx = \lambda\) with \(I^*(u) = I_\lambda^* < \infty\). (2.15) is valid, for instance, when \(h(x) < \bar{h}\) for any \(x \in \mathbb{R}^n\), or \(a_{\alpha\beta}(x, u) \xi_\alpha \xi_\beta < \bar{a}_{\alpha\beta}(u) \xi_\alpha \xi_\beta\) for any \((x, u, \xi) \in \mathbb{R}^n \times (\mathbb{R}^N - \{0\}) \times (\mathbb{R}^n - \{0\})\).

**Example 2.1.** In (1.1), if \(n = 3, p = 5\), \(h(x)\) satisfies (v), and

\[
a_{\alpha\beta}(x, u) = (1 + |u|^2) b_{\alpha\beta}(x) \quad \text{(or, } a_{\alpha\beta}(x, u) = b_{\alpha\beta}(x)/(1 + |u|^2))
\]

where \(b_{\alpha\beta}(x) \in C^1(\mathbb{R}^n)\) and \(b_{\alpha\beta} = b_{\beta\alpha} (1 \leq \alpha, \beta \leq n)\) satisfy

\[0 < \lambda |\xi|^2 \leq b_{\alpha\beta}(x) \xi_\alpha \xi_\beta \leq M |\xi|^2\]

for any \((x, \xi) \in \mathbb{R}^n \times \mathbb{R}^n\) where \(\lambda, M > 0\) are constants, and \(\lim_{|x| \to \infty} b_{\alpha\beta}(x) = \bar{b}_{\alpha\beta}\), then it is easy to see that \(a_{\alpha\beta}(x, u)\), \(h(x)\) satisfy conditions (i)-(vii), and
thus we conclude that (1.1) possesses at least a nontrivial weak solution by using Theorem 2.3.

The above is only a simple example, the theorems in this section are applicable to many other cases.

3. Proof of Theorems 2.1 and 2.2

In this section, we prove Theorem 2.1 and Theorem 2.2. We need some lemmata and we always suppose that conditions (i)-(v) hold in this section.

Lemma 3.1. $I_{\lambda}, I_{\lambda}^{s}$ are continuous functions of $\lambda$ on $[0, +\infty)$.

Proof. It is evident that $I_{\lambda}, I_{\lambda}^{s}$ are all finite for each $\lambda \geq 0$. Let $\lambda_{m} \to \lambda_{0} \in (0, +\infty)$. We may assume that $\lambda_{m} > 0$ for any $m > 0$. Given $\epsilon > 0$ we have by (2.13), that there are $(u_{m}) \subset E$ such that $\int_{\mathbb{R}^{n}}|u_{m}|^{p}dx = 1$ and

$$I(\lambda_{m}^{1/p}u_{m}) \leq I_{\lambda_{0}} + \epsilon.$$  

We claim that $|I_{\lambda_{m}}| \leq C$ (hereafter $C$ denotes a constant independent of $m$). In fact, for fixed $u_{0} \in C_{0}^{\infty} \subset E$ with $\int_{\mathbb{R}^{n}}|u_{0}|^{p}dx = 1$, we have by (2.1), the fact that $|\lambda_{m}| \leq C$ and the continuity of $\sigma(t)$, that

$$I_{\lambda_{m}} \leq I(\lambda_{m}^{1/p}u_{0}) = \lambda_{m}^{2/p} \int_{\mathbb{R}^{n}}[a_{\alpha\beta}(x,\lambda_{m}^{1/p}u_{0})D_{\alpha}u_{0}D_{\beta}u_{0} + h(x)|u_{0}|^{2}]dx$$

$$\leq \lambda_{m}^{2/p} \int_{\mathbb{R}^{n}}\sigma(|\lambda_{m}^{1/p}u_{0}|)|Du_{0}|^{2} + \lambda_{m}^{2/p} \int_{\mathbb{R}^{n}}h(x)|u_{0}|^{2}dx \leq C < +\infty.$$  

Hence, by (i) we get

$$\int_{\mathbb{R}^{n}}[\sigma(\lambda_{m}^{1/p}|u_{m}|)|Du_{m}|^{2} + h(x)|u_{m}|^{2}]dx \leq I_{\lambda_{m}} + \epsilon \leq C.$$  

Since $\sigma(t)$ is nondecreasing in $t$, it is trivial that

$$\int_{\mathbb{R}^{n}}[\sigma(\lambda_{0}^{1/p}|u_{m}|)|Du_{m}|^{2} + h(x)|u_{m}|^{2}]dx \leq C$$  

when $\lambda_{m} \geq \lambda_{0}$, while if $\lambda_{m} < \lambda_{0}$, we have by (2.2) and the boundedness of $(\lambda_{0}/\lambda_{m})^{1/p}$, that

$$\int_{\mathbb{R}^{n}}[\sigma(\lambda_{0}^{1/p}|u_{m}|)|Du_{m}|^{2} + h(x)|u_{m}|^{2}]dx$$

$$= \int_{\mathbb{R}^{n}}\left[\sigma\left(\frac{\lambda_{0}}{\lambda_{m}}\right)^{1/p}|\lambda_{m}^{1/p}|u_{m}|)|Du_{m}|^{2} + h(x)|u_{m}|^{2}\right]dx$$

$$\leq C_{m} \int_{\mathbb{R}^{n}}[\sigma(\lambda_{m}^{1/p}|u_{m}|)|Du_{m}|^{2} + h(x)|u_{m}|^{2}]dx \leq C < +\infty.$$
Thus, we always have

$$
\int_{\mathbb{R}^n} \left[ \sigma(\lambda_0^{1/p}|u_m|)|Du_m|^2 + h(x)|u_m|^2 \right] \, dx \leq C.
$$

It is clear that

$$
I_m + \epsilon \geq I(\lambda_m^{1/p} u_m) = I(\lambda_m^{1/p} u_m) - I(\lambda_0^{1/p} u_m) + I(\lambda_0^{1/p} u_m) - I(\lambda_0^{1/p} u_m) + I_0,
$$

but

$$
I(\lambda_m^{1/p} u_m) - I(\lambda_0^{1/p} u_m) = \lambda_m^{2/p} \int_{\mathbb{R}^n} [a_{\alpha\beta}(x, \lambda_m^{1/p} u_m) - a_{\alpha\beta}(x, \lambda_0^{1/p} u_m)]D_\alpha u_m^\alpha D_\beta u_m^\beta \, dx

+ (\lambda_m^{2/p} - \lambda_0^{2/p}) \int_{\mathbb{R}^n} a_{\alpha\beta}(x, \lambda_0^{1/p} u_m)D_\alpha u_m^\alpha D_\beta u_m^\beta \, dx

+ (\lambda_m^{2/p} - \lambda_0^{2/p}) \int_{\mathbb{R}^n} h(x)|u_m|^2 \, dx

\equiv I_m^1 + I_m^2 + I_m^3.
$$

It is trivial that \(\lim_{m \to \infty} I_m^1 = 0\) and by (2.1) and (3.2) we have that \(\lim_{m \to \infty} I_m^2 = 0\). On the other hand, by the mean value theorem, we have

$$
|I(\lambda_m^{1/p} u_m) - I(\lambda_0^{1/p} u_m)| = |(\lambda_m^{1/p} - \lambda_0^{1/p}) u_m^\alpha D_\alpha a_{\alpha\beta}(x, \xi_m(x) u_m)D_\beta u_m^\beta|,
$$

where \(\xi_m(x)\) is between \(\lambda_m^{1/p}\) and \(\lambda_0^{1/p}\), hence \(|\xi_m(x)| \geq C > 0\). So, by (2.6), (3.1) and (3.2) we have

$$
\left| \int_{\mathbb{R}^n} u_m^\alpha D_\alpha a_{\alpha\beta}(x, \xi_m(x) u_m)D_\alpha u_m^\alpha D_\beta u_m^\beta \, dx \right| \leq C \int_{\mathbb{R}^n} \sigma(\xi_m(x)|u_m|)|Du_m|^2 \, dx

\leq \max_{0 \leq m} C \int_{\mathbb{R}^n} \sigma(\lambda_m^{1/p}|u_m|)|Du_m|^2 \, dx

\leq C
$$

from which \(\lim_{m \to \infty} I_m^3 = 0\) and hence \(\liminf_{m \to \infty} I_m = \epsilon \geq I_0\). Thus we have \(\liminf_{m \to \infty} I_m \geq I_0\), which shows that \(I_\lambda\) is lower-semi continuous on \((0, +\infty)\). On the other hand, it is trivial to see that \(\limsup_{m \to \infty} I_m \leq I_0\), which gives that \(I_\lambda\) is upper-semi continuous on \((0, +\infty)\). So we see that \(I_\lambda\) is continuous on \((0, +\infty)\). It is trivial that \(I_\lambda\) is continuous at \(\lambda = 0\) and the lemma is proved. \(\square\)

**Lemma 3.2.** For any \(\lambda > 0\), we have
(3.3) \[ I_\lambda \leq I_\lambda^\alpha \]
(3.4) \[ I_\lambda^\alpha < I_\lambda + I_{\lambda - \alpha}^\alpha \quad \text{for every } \alpha \in (0, \lambda) \]
(3.5) \[ I_\lambda < I_\lambda + I_{\lambda - \alpha}^\alpha \quad \text{for every } \alpha \in (0, \lambda) \]

If \( I_\beta < I_\beta^\alpha \) for any \( \beta > 0 \), then

(3.6) \[ I_\lambda < I_\lambda + I_{\lambda - \alpha}^\alpha \quad \text{for every } \alpha \in [0, \lambda). \]

**Proof.** By (iv) and (v), it is trivial that (3.3) holds. To prove (3.5), we only need to show that

(3.7) \[ I_{\theta \gamma} < \theta I \quad \text{for every } \gamma \in (0, \lambda), \theta \in \left(1, \frac{\lambda}{\gamma}\right) \]

(see Lemma II.1 of [4]). Given \( \gamma \in (0, \lambda), \theta \in \left(1, \frac{\lambda}{\gamma}\right) \), we have by (2.13) and (2.4), that

\[
I_{\theta \gamma} = (\theta \gamma)^{2/p} \inf \left\{ \int_{\Omega^n} \left[ a_{\alpha \beta}(x, \theta \gamma^{1/p} u) D_\alpha u D_\beta u' + h(x)|u|^2 \right] \, dx : u \in E, \int_{\Omega^n} |u|^p \, dx = 1 \right\}
\]

\[
\leq \theta^{2/p} \gamma^{2/p} \inf \left\{ \int_{\Omega^n} \left[ a_{\alpha \beta}(x, \gamma^{1/p} u) D_\alpha u D_\beta u' + h(x)|u|^2 \right] \, dx : u \in E, \int_{\Omega^n} |u|^p \, dx = 1 \right\}
\]

\[
= \theta^{(2 + \gamma)/p} I_{\gamma} < \theta I_{\gamma}
\]

Here we have made use of \( I_{\gamma} > 0 \) (for all \( \gamma > 0 \)) which can easily be derived from the definition. Thus (3.7) holds and hence (3.5) holds. Similarly, by Remark 2.1 we see that (3.4) holds. By (3.3), (3.5) and \( I_\beta < I_\beta^\alpha \) (for all \( \beta > 0 \)), we see that (3.6) holds. \( \square \)

**Proof of Theorem 2.1 and Theorem 2.2.** Let \( (u_m) \subset E \) be a minimizing sequence of \( I_\lambda \) (or \( I_\lambda^\alpha \)) with

\[
\int_{\Omega^n} |u_m|^p \, dx = \lambda > 0
\]

and

\[
I(u_m) < I_\lambda + 1/m \quad \text{(or } I_\lambda^\alpha(u_m) < I_\lambda + 1/m). \]

Since \( I_\lambda \) is finite, by (ii) we have

(3.8) \[ \int_{\Omega^n} \left[ \sigma(|u_m|)|D u_m|^2 + h(x)|u_m|^2 \right] \, dx \leq C \]
(or)
\[ \int_{\mathbb{R}^n} |\sigma(|u_m|)|D u_m|^2 + \beta |u_m|^2 \, dx \leq C \]

in the case of \( I_{\infty}^\alpha \) and \( |u_m| \leq C \).

By the Sobolev embedding theorem, we may assume the existence of a \( u_0 = (u_0^1, u_0^2, \ldots, u_0^N) \in E \) such that

\[ \begin{align*}
  u_m &\rightharpoonup u_0 \quad \text{in} \quad E \\
  u_m^i &\rightarrow u_0^i \quad \text{in} \quad H^1 \left( \mathbb{R}^n \right), \quad 1 \leq i \leq N \\
  u_m &\rightarrow u_0 \quad \text{a.e. in} \quad \mathbb{R}^n \\
  u_m^i &\rightarrow u_0^i \quad \text{in} \quad L^t_{\text{loc}} \left( \mathbb{R}^n \right), \quad 2 \leq t < \frac{2n}{n-2}
\end{align*} \]  

where \( \rightharpoonup \) designates weak convergence, while \( \rightarrow \) means strong convergence.

Let
\[ \rho_m = a_{\alpha \beta}(x, u_m) D_{\alpha} u_m^i D_{\beta} u_m^i + h(x) |u_m|^2 \]

(respectively
\[ \rho_m = \bar{a}_{\alpha \beta}(u_m) D_{\alpha} u_m^i D_{\beta} u_m^i + \beta |u_m|^2 \]

in the case of \( I_{\infty}^\alpha \), and
\[ \lambda_m = \int_{\mathbb{R}^n} \rho_m \, dx, \]

we easily see that \( \lambda_m \geq C > 0 \). We need the following concentration compactness lemma:

**Lemma 3.3.** Let \( u_m, \rho_m, \lambda_m \) be as above, then there exists a subsequence of \( (\rho_m) \), still denoted by \( (\rho_m) \), satisfying one of the three following possibilities:

(i) **(Compactness)** There exists \( y_m \in \mathbb{R}^n \) such that \( \rho_m(x + y_m) \) is tight, i.e. for every \( \varepsilon > 0 \), there exists \( R \) such that
\[ \int_{y_m + B_R} \frac{\rho_m(x)}{\lambda_m} \, dx \geq 1 - \varepsilon, \]

where
\[ y_m + B_R = \{ x \in \mathbb{R}^n : |x - y_m| \leq R \}. \]

(ii) **(Vanishing)** \( \lim_{m \to \infty} \sup_{y \in \mathbb{R}^n} \int_{y + B_R} \rho_m(x) \, dx = 0 \) for all \( R < + \infty \).
(iii) (Dichotomy) There exist \( \alpha \in (0, 1) \) and a positive function \( \mu(\epsilon) \), with 
\[
\lim_{\epsilon \to 0} \mu(\epsilon) = 0,
\]
such that for every \( \epsilon > 0 \) there exist \( m_0 \geq 1 \) and \( u_m^1, u_m^2 \in E \)
with \( |u_m^1|, |u_m^2| \leq C \), so that

\[
\lim_{m \to \infty} \text{dist} (\text{supp} \ u_m^1, \text{supp} \ u_m^2) = +\infty
\]

\[
|u_m - (u_m^1 + u_m^2)|_2 \leq \mu(\epsilon)
\]

\[
|u_m - (u_m^1 + u_m^2)|_p < \mu(\epsilon)
\]

\[
\left| \frac{I(u_m^1)}{\lambda_m} - \alpha \right| < \mu(\epsilon)
\]

\[
\left| \frac{I(u_m^2)}{\lambda_m} - (1 - \alpha) \right| < \mu(\epsilon)
\]

\[
I(u_m^1) \geq I(u_m^1) + I(u_m^2) - \mu(\epsilon)
\]

or, respectively, in the case of \( I^\infty \),

\[
\left| \frac{I^\infty(u_m^1)}{\lambda_m} - \alpha \right| < \mu(\epsilon)
\]

\[
\left| \frac{I^\infty(u_m^2)}{\lambda_m} - (1 - \alpha) \right| < \mu(\epsilon)
\]

\[
I^\infty(u_m^1) \geq I^\infty(u_m^1) + I^\infty(u_m^2) - \mu(\epsilon).
\]

**Proof.** For any \( t \geq 0 \), let

\[
Q_m(t) = \sup_{y \in B_1} \int_{y + B_r} \frac{\rho_m}{\lambda_m} \, dx.
\]

Then \( Q_m(t) \) is nondecreasing in \( t \) and \( |Q_m(t)| \leq 1 \), so by Helly's principle there
is a subsequence of \( Q_m(t) \), still denoted by \( Q_m(t) \) with
\( \lim_{m \to \infty} Q_m(t) = Q(t) \)
for any \( t \geq 0 \), where \( Q(t) \) is a nondecreasing function on \( [0, +\infty) \) and
\( |Q(t)| \leq 1 \).

Let \( \lim_{t \to +\infty} Q(t) = \alpha \in [0, 1] \). If \( \alpha = 0 \), then \( Q(t) \equiv 0 \), hence \( \lim_{m \to \infty} Q_m(t) = 0 \) and (ii) (vanishing) occurs.

If \( \alpha = 1 \), we can easily show that (i) (compactness) occurs by using the same
method as in the proof of Lemma I.1 of [4].

Now, letting \( \alpha \in (0, 1) \), we want to show that (iii) (dichotomy) occurs.

Given \( \epsilon > 0 \), there exists \( R_0 = R_0(\epsilon) > 0 \) such that

\[
\alpha - \epsilon < Q(R_0) < \alpha + \epsilon
\]

\[
\alpha - 2\epsilon < Q(2R_0) < \alpha + 2\epsilon
\]
hence there exists $m_0(\epsilon) > 0$ with

\begin{align}
(3.19) & \\ 
\alpha - \epsilon < Q_m(R_0) < \alpha + \epsilon \\
(3.20) & \\ 
\alpha - 2\epsilon < Q_m(2R_0) < \alpha + 2\epsilon
\end{align}

whenever $m \geq m_0$.

We may choose $R_m \to +\infty$ such that

\begin{equation}
Q_m(2R_m) < \alpha + 1/m.
\end{equation}

By the absolute continuity of Lebesgue integrals, there are $(z_m) \subset \mathbb{R}^n$ such that

\begin{equation}
Q_m(R_0) = \int_{z_m + B_{R_0}} \frac{\rho_m}{\lambda_m} \, dx.
\end{equation}

Let $\xi, \varphi \in C_c^\infty(\mathbb{R}^n)$, $0 \leq \xi$, $\varphi \leq 1$, $\xi = 1$ and $\varphi = 0$ if $|x| \leq 1$; $\xi = 0$ and $\varphi = 1$ if $|x| \geq 2$ and set $\xi_m = \xi[(x - z_m)/\tilde{R}]/\tilde{R}$ ($\tilde{R} \geq R_0$ is to be determined) $\varphi_m = \varphi[(x - 3m)/R_m]$ and $u_m^0 = \xi_m u_m$, $u_m^1 = \varphi_m u_m$. It is trivial that (3.10) holds and that $|u_m^0|$, $|u_m^1| \leq C$.

By (3.22) we have

\begin{equation}
Q_m(R_0) = \frac{1}{\lambda_m} \int_{z_m + B_{R_0}} \left[ a_{\alpha\beta}(x, u_m) D_\alpha u_m^0 D_\beta u_m^1 + h(x)|u_m|^2 \right] \, dx
\end{equation}

\begin{align}
&= \frac{1}{\lambda_m} \int_{z_m + B_{R_0}} \left[ a_{\alpha\beta}(x, \xi_m u_m) D_\alpha (\xi_m u_m^0) D_\beta (\xi_m u_m^1) \\
&\quad + h(x)|\xi_m u_m|^2 \right] \, dx \\
&= \frac{1}{\lambda_m} I(u_m^1)
\end{align}

\begin{equation}
- \frac{1}{\lambda_m} \int_{|x - z_m| \approx R_0} \left[ a_{\alpha\beta}(x, u_m^0) D_\alpha (u_m^0)^1 D_\beta (u_m^1)^1 \\
+ h(x)|u_m^0|^2 \right] \, dx
\end{equation}

We want to show that

\begin{equation}
\frac{1}{\lambda_m} \int_{|x - z_m| \approx R_0} \left[ a_{\alpha\beta}(x, u_m^0) D_\alpha (u_m^0)^1 D_\beta (u_m^1)^1 + h(x)|u_m^0|^2 \right] \, dx < \mu(\epsilon).
\end{equation}

Since

\begin{equation}
\frac{1}{\lambda_m} \int_{|x - z_m| \approx R_0} \left[ a_{\alpha\beta}(x, u_m) D_\alpha (u_m^0)^1 D_\beta (u_m^1)^1 + h(x)|u_m^0|^2 \right] \, dx
\end{equation}

\begin{align}
\leq \frac{1}{\lambda_m} \int_{R_0 \leq |x - z_m| \leq 2R} \left[ a_{\alpha\beta}(x, u_m) (u_m^0 D_\alpha \xi_m + \xi_m D_\alpha u_m^0)(u_m^1 D_\beta \xi_m + \xi_m D_\beta u_m^1) \\
+ h(x)|u_m|^2 \right] \, dx
\end{align}
\[
\int_{R_0 \leq |x - z_m| \leq 2R} \xi_m^2 \alpha_{\alpha\beta}(x, u_m^i) D_{\alpha} u_m^i D_{\beta} u_m^i \, dx \\
+ \frac{2}{\lambda_m} \int_{R_0 \leq |x - z_m| \leq 2R} \xi_m u_m^i \alpha_{\alpha\beta}(x, u_m^i) D_{\alpha} \xi_m D_{\beta} u_m^i \, dx \\
+ \frac{1}{\lambda_m} \int_{R_0 \leq |x - z_m| \leq 2R} a_{\alpha\beta}(x, u_m^i) D_{\alpha} \xi_m D_{\beta} \xi_m \cdot u_m^i u_m^i \, dx \\
+ \frac{1}{\lambda_m} \int_{R_0 \leq |x - z_m| \leq 2R} b(x) |u_m|^2 \, dx \\
= J_m^1 + J_m^2 + J_m^3 + J_m^4.
\]

By (3.19), (3.21) and the fact that \( Q_m(t) \) is nondecreasing, it is evident that

\[
|J_m^4| \leq Q_m(2R) - Q_m(R_0) < \alpha + 1/m - (\alpha - \varepsilon) = 1/m + \varepsilon < \mu(\varepsilon)
\]

(for \( m \) large enough).

By (2.1), (2.2) and (2.3) and since \( |u_m| \leq C \), we have that

\[
|J_m^3| \leq 2a_3 \int_{R_0 \leq |x - z_m| \leq 2R} \sigma(|\xi_m u_m|) |D\xi_m|^2 |u_m|^2 \, dx
\]

\[
\leq \frac{C}{R^2} \int_{R_0 \leq |x - z_m| \leq 2R} \sigma(|u_m|) |u_m|^2 \, dx
\]

\[
\leq \frac{C}{R^2} \int_{\mathbb{R}^n} (|u_m|^2 + |u_m|^{q+2}) \, dx
\]

\[
\leq \frac{C}{R^2} < \mu(\varepsilon),
\]

for \( R(\varepsilon) \) large enough. In the same way, using (2.3) and (3.8) we have that

\[
|J_m^2| \leq \frac{C}{R} \int_{R_0 \leq |x - z_m| \leq 2R} |a_{\alpha\beta}(x, \xi_m u_m) D_{\alpha} u_m^i D_{\beta} u_m^i| \, dx
\]

\[
\leq \frac{C}{R} \int_{\mathbb{R}^n} \sigma(|u_m|) |Du_m| |u_m| \]

\[
\leq \frac{C}{R} \int_{\mathbb{R}^n} \sigma(|u_m|) (|Du_m|^2 + |u_m|^2) \, dx
\]

\[
< \frac{C}{R} < \mu(\varepsilon)
\]
for $R(\epsilon)$ large enough. By (2.1), (3.19), (3.21) and (3.22) we have that

\[
0 \leq J_m^1 \leq C \int_{R_0 \leq |x-z_m| \leq 2R} \sigma(|u_m|)|Du_m|^2 \\
\leq C \int_{R_0 \leq |x-z_m| \leq 2R} a_{\alpha\beta}(x,u_m)D_{\alpha}u_m^i D_{\beta}u_m^i \\
\leq Q_m(2R_m) - Q_m(R_0) < \alpha + 1/m - (\alpha - \epsilon) \\
= 1/m + \epsilon < \mu(\epsilon)
\]

(for $m$ large enough).

Combining the above estimates, we see that (3.24) holds and (3.13) holds by (3.23). Similarly, (3.16) holds.

It is easy to show (see e.g. Lemma 1.1 of [4]) that

\[
(3.26) \left| \int_{|x-z_m| \leq 2R_m} \frac{1}{\lambda_m} [a_{\alpha\beta}(x,u_m)D_{\alpha}u_m^i D_{\beta}u_m^i + h(x)|u_m|^2] \, dx - (1 - \alpha) \right| < \mu(\epsilon)
\]

On the other hand, we have

\[
\frac{1}{\lambda_m} I(u_m^2) = \frac{1}{\lambda_m} \int_{|x-z_m| \leq R_m} [a_{\alpha\beta}(x,u_m^2)D_{\alpha}(u_m^2)^i D_{\beta}(u_m^2)^i + h(x)|u_m^2|^2] \, dx \\
= \frac{1}{\lambda_m} \int_{R_m \leq |x-z_m| \leq 2R_m} [a_{\alpha\beta}(x,u_m^2)D_{\alpha}(u_m^2)^i D_{\beta}(u_m^2)^i + h(x)|u_m^2|^2] \, dx \\
+ \frac{1}{\lambda_m} \int_{|x-z_m| \leq 2R_m} [a_{\alpha\beta}(x,u_m)D_{\alpha}u_m^i D_{\beta}u_m^i + h(x)|u_m|^2] \, dx
\]

(3.27)

Similarly to (3.24), we can prove that

\[
(3.28) \frac{1}{\lambda_m} \int_{R_m \leq |x-z_m| \leq 2R_m} [a_{\alpha\beta}(x,u_m^2)D_{\alpha}(u_m^2)^i D_{\beta}(u_m^2)^i + h(x)|u_m^2|^2] \, dx \leq \mu(\epsilon)
\]

Thus (3.26) and (3.27) imply that (3.14) holds. Similarly, (3.17) holds.

By (3.19) and (3.21) we have that

\[
\|u_m - (u_m^1 + u_m^2)\|^2 = \int_{\mathbb{R}^n} |1 - \xi_m - \varphi_m|^2 |u_m|^2 \, dx \\
\leq C \int_{|x-z_m| \leq 2R_m} |u_m|^2 \\
\leq C(Q_m(2R_m) - Q_m(R_0)) < \mu(\epsilon).
\]
So we have (3.11). Similarly, by \( \| u_m \| \leq C \) and \( \| u_m^1 \| \leq C, \| u_m^2 \| \leq C \), we see that (3.12) holds.

Finally we prove (3.15). Since

\[
I(u_m) \geq \int_{|x - z_m| \leq R} \left[ a_{\alpha \beta}(x, u_m) D_\alpha u_m^i D_\beta u_m^i + h(x) |u_m|^2 \right] dx
\]

\[+ \int_{|x - z_m| = 2R_m} \left[ a_{\alpha \beta}(x, u_m) D_\alpha u_m^i D_\beta u_m^i + h(x) |u_m|^2 \right] dx \]

\[= I(u_m^1) + I(u_m^2) \]

\[\int_{R_m \leq |x - z_m| \leq 2R_m} \left[ a_{\alpha \beta}(x, u_m^2) D_\alpha (u_m^2)^i D_\beta (u_m^2)^i + h(x) |u_m^2|^2 \right] dx \]

and because of (3.24) and (3.28), we deduce that

\[I(u_m) \geq I(u_m^1) + I(u_m^2) - \mu(\epsilon).\]

Thus (3.15) holds. Similarly (3.18) holds. \( \square \)

Lemma 3.4. (cf. Lemma 1.1 of [5].) Let \( 1 < p \leq \infty, \ 1 \leq q < \infty, \) with \( q \neq Np/(N - p) \) if \( p < N \). Assume that \( (u_m) \) is bounded in \( L^q(\mathbb{R}^N) \), \( |Du_m| \) is bounded in \( L^p(\mathbb{R}^N) \) and

\[\sup_{y \in \mathbb{R}^N} \int_{y + B_R} |u_m|^q dx \to 0 \quad \text{as} \quad m \to \infty, \quad \text{for some} \quad R > 0.\]

Then \( u_m \to 0 \) in \( L^r(\mathbb{R}^N) \) for any \( r \) between \( q \) and \( Np/(N - p) \).

We now turn to prove Theorem 2.1 and Theorem 2.2. We already know that there is a minimizing sequence \( (u_m) \) of \( I_\lambda \) (or \( I_\lambda^c \)) such that Lemma 3.3 holds.

If «vanishing» occurs, then

\[\lim_{m \to \infty} \sup_{y \in \mathbb{R}^N} \int_{y + B_R} \left[ a_{\alpha \beta}(x, u_m) D_\alpha u_m^i D_\beta u_m^i + h(x) |u_m|^2 \right] dx = 0 \]

(3.29)

for all \( R \). We know also that \( (Du_m) \) is bounded in \( L^2(\mathbb{R}^n) \) and by (3.29) we know that

\[\lim_{m \to \infty} \sup_{y \in \mathbb{R}^N} \int_{y + B_R} |u_m|^2 dx = 0 \quad \text{(for any} \ R > 0).\]
So Lemma 3.4 gives that
\[
\lim_{m \to +\infty} \int_{\mathbb{R}^n} u_m^p \, dx = 0
\]
and this contradicts
\[
\int_{\mathbb{R}^n} |u_m|^p \, dx = \lambda.
\]
Thus we have ruled out «vanishing».

If «dichotomy» occurs, then Lemma 3.3 shows that for any \( \epsilon > 0 \), there are \( u_m^+, u_m^- \in E \) such that (3.10)-(3.15) hold (or (3.10), (3.12), (3.5) and (3.18) hold in the case of \( I_k^n \)). Therefore we would have that
\[
I_k + \epsilon \geq I(u_m) \\
\geq I(u_m^+) + I(u_m^-) - \mu(\epsilon) \\
\geq I\left(\int_{\mathbb{R}^n} |u_m^+|^p \, dx + I\left(\int_{\mathbb{R}^n} |u_m^-|^p \, dx - \mu(\epsilon)\right).
\]

We may assume that
\[
\lim_{m \to +\infty} \int_{\mathbb{R}^n} |u_m^+|^p \, dx = \lambda_1(\epsilon), \quad \lim_{m \to +\infty} \int_{\mathbb{R}^n} |u_m^-|^p \, dx = \lambda_2(\epsilon).
\]

Now
\[
\lambda = \int_{\mathbb{R}^n} |u_m|^p \, dx
\]
and
\[
\left| \int_{\mathbb{R}^n} |u_m|^p \, dx - \int_{\mathbb{R}^n} |u_m^+|^p \, dx - \int_{\mathbb{R}^n} |u_m^-|^p \, dx \right| \leq \int_{\mathbb{R}^n} \left| 1 - \phi_m - \xi_m^p \right| |u_m|^p \, dx \\
\leq C \int_{R_0} \int_{|x - z_m| \leq 2R_m} |u_m|^p \, dx \\
\leq C \left( \int_{R_0} \int_{|x - z_m| \leq 2R_m} |u_m|^2 \, dx \right)^{p/2} \\
< \mu(\epsilon),
\]
(where we have made use of notations in the proof of Lemma 3.3.)

We conclude that
\[
|\lambda - (\lambda_1(\epsilon) + \lambda_2(\epsilon))| \leq \mu(\epsilon)
\]
Letting \( m \to \infty \) in (3.30) and using Lemma 3.1 we obtain that
\[
I_k + \epsilon \geq I_{\lambda_1(\epsilon)} + I_{\lambda_2(\epsilon)} - \mu(\epsilon).
\]
We assume now that $\lambda_1(\varepsilon) \to \lambda_1$, $\lambda_2(\varepsilon) \to \lambda_2$ as $\varepsilon \to 0$. Then we have by Lemma 3.1, that

\begin{equation}
(3.32) \quad I_\lambda \geq I_{\lambda_1} + I_{\lambda_2}.
\end{equation}

By Lemma 3.3 and the fact that $\lambda_m \geq c > 0$ we have that

\begin{align*}
|I(u_m^1) - \tilde{u}| < \mu(\varepsilon), \quad &\text{where} \quad \tilde{u} > 0 \\
|I(u_m^2) - \beta| < \mu(\varepsilon), \quad &\text{where} \quad \beta > 0.
\end{align*}

Thus, if $\lambda_1 = 0$ then by (3.31) $\lambda_2 = \lambda$. Since

\begin{equation*}
I_\lambda + \varepsilon \geq I(u_m) \geq I(u_m^1) + I(u_m^2) - \mu(\varepsilon)
\end{equation*}

we obtain that

\begin{equation*}
I_\lambda \geq \tilde{u} + I_{\lambda_2} - \mu(\varepsilon).
\end{equation*}

Hence

\begin{equation*}
I_\lambda \geq \tilde{u} + I_\lambda.
\end{equation*}

This is a contradiction and so $\lambda_1 > 0$; similarly $\lambda_2 > 0$. And now $\lambda_1 + \lambda_2 = \lambda$ and (3.32) contradict (3.5). Thus we have ruled out the «dichotomy» for $I_\lambda$. Similarly we can rule out the «dichotomy» for $I^m_\lambda$ using (3.4).

So we only have «compactness» i.e. there exists $(y_m) \subset \mathbb{R}^n$ such that for any $\varepsilon > 0$ there exists $R = R(\varepsilon) > 0$ with

\begin{equation*}
\int_{|x - y_m| \geq R} [a_{ij}(x, u_m)D_{ij}u_m + h(x)|u_m|^2] \, dx \leq \lambda_m(1 - \varepsilon).
\end{equation*}

Hence

\begin{equation}
(3.33) \quad \int_{|x - y_m| \geq R} [a_{ij}(x, u_m)D_{ij}u_m + h(x)|u_m|^2] \, dx \leq \lambda_m \varepsilon
\end{equation}

or, in the case of $I^m_\lambda$, we have

\begin{equation}
(3.34) \quad \int_{|x - y_m| \geq R} [a_{ij}(u_m)D_{ij}u_m + h|u_m|^2] \, dx \leq \lambda_m \varepsilon
\end{equation}

We first prove Theorem 2.1. Let $\tilde{u}_m(x) = u_m(x + y_m)$, then $|\tilde{u}_m| \leq C < +\infty$ and by (3.34) and the Sobolev embedding theorem we may assume the existence of a $u = (u^1, u^2, \ldots, u^N) \in E$ such that
\begin{align}
\begin{cases}
\bar{u}_m \to u & \text{in } E \\
\bar{u}^i_m \to u^i & \text{in } H^1(R^n) \\
\bar{u}^t_m \to u^t & \text{in } L^t(R^n) \quad 2 \le t < 2\hat{\eta}/(\hat{\eta} - 2) \\
\bar{u}_m \to u & \text{a.e. in } R^n
\end{cases}
\end{align}

for 1 \le i \le N, and

\[\lambda = \int_{R^n} |u_m|^p \, dx = \int_{R^n} |\bar{u}_m|^p \, dx \to \int_{R^n} |u|^p \, dx \quad \text{(as } m \to \infty).\]

Also

\[I^*_\lambda = \lim_{m \to \infty} \int_{R^n} [\bar{a}_{\alpha\beta}(\bar{u}_m)D_\alpha \bar{u}_m D_\beta \bar{u}_m + \bar{h}|u_m|^2] \, dx.\]

By (3.35) and (ii), (iii) of Section 2 we see that

\[\bar{a}_{\alpha\beta}(\bar{u}_m) \to \bar{a}_{\alpha\beta}(u) \quad \text{a.e. in } R^n.\]

So for any bounded domain \(\Omega \subset R^n\) and \(\delta > 0\), there is a \(\Omega_\delta \subset \Omega\) with

\[|\Omega - \Omega_\delta| < \delta\]

and

\[\bar{a}_{\alpha\beta}(\bar{u}_m) \to \bar{a}_{\alpha\beta}(u)\]

uniformly for \(x \in \Omega_\delta\) where \(|A|\) denotes the Lebesgue measure of \(A\) for any \(A \subset R^n\). So that for any \(\epsilon > 0\) and \(m\) large enough we have, by (2.2), that

\[\int_{\Omega_\delta} \bar{a}_{\alpha\beta}(\bar{u}_m)D_\alpha \bar{u}_m D_\beta \bar{u}_m \, dx \ge \int_{\Omega_\delta} \bar{a}_{\alpha\beta}(\bar{u}_m)D_\alpha \bar{u}_m D_\beta \bar{u}_m \, dx \]

\[\ge \int_{\Omega_\delta} [\bar{a}_{\alpha\beta}(\bar{u}_m) - \bar{a}_{\alpha\beta}(u)]D_\alpha \bar{u}_m D_\beta \bar{u}_m \, dx \]

\[+ \int_{\Omega_\delta} \bar{a}_{\alpha\beta}(u)D_\alpha \bar{u}_m D_\beta \bar{u}_m \, dx \]

\[\ge -\epsilon \int_{\Omega_\delta} |D\bar{u}_m|^2 \, dx + \int_{\Omega_\delta} \bar{a}_{\alpha\beta}(u)D_\alpha \bar{u}_m D_\beta \bar{u}_m \, dx \]

\[\ge -\epsilon C + \int_{\Omega_\delta} \bar{a}_{\alpha\beta}(u)D_\alpha \bar{u}_m D_\beta \bar{u}_m \, dx.\]

By (3.35), Mazur's theorem (see [6]) and Fatou's lemma, we see that

\[\liminf_{m \to \infty} \int_{\Omega_\delta} \bar{a}_{\alpha\beta}(u)D_\alpha \bar{u}_m D_\beta \bar{u}_m \, dx \ge \int_{\Omega_\delta} \bar{a}_{\alpha\beta}(u)D_\alpha u^t D_\beta u^t \, dx,\]
and hence we get, for any $N$, that

$$
\liminf_{m \to \infty} \int_{\Omega} \bar{a}_{\alpha \beta}(\bar{u}_m) D_{\alpha} \bar{u}_m D_{\beta} \bar{u}_m \, dx \geq \int_{\Omega} \bar{a}_{\alpha \beta}(u) D_{\alpha} u D_{\beta} u \, dx
$$

$$
\geq \int_{\Omega} [\bar{a}_{\alpha \beta}(u) D_{\alpha} u D_{\beta} u]_N \, dx
$$

where the function $[f]_N$ for any $f \geq 0$ is given by

$$
[f]_N = \begin{cases} 
    f & \text{if} \quad f \leq N \\
    N & \text{if} \quad f \geq N
\end{cases}
$$

Since $[\bar{a}_{\alpha \beta}(u) D_{\alpha} u D_{\beta} u]_N \in L^1(\Omega)$, and since $|\Omega| \to |\Omega|$ we have that

$$
\liminf_{m \to \infty} \int_{\Omega} \bar{a}_{\alpha \beta}(\bar{u}_m) D_{\alpha} \bar{u}_m D_{\beta} \bar{u}_m \, dx \geq \int_{\Omega} [\bar{a}_{\alpha \beta}(u) D_{\alpha} u D_{\beta} u]_N \, dx.
$$

Letting $N \to \infty$, we have that

(3.36) \quad \liminf_{m \to \infty} \int_{\Omega} \bar{a}_{\alpha \beta}(\bar{u}_m) D_{\alpha} \bar{u}_m D_{\beta} \bar{u}_m \, dx \geq \int_{\Omega} \bar{a}_{\alpha \beta}(u) D_{\alpha} u D_{\beta} u \, dx

Thus, since the supremum of any sequence of lower-semicontinuous functions is still lower-semicontinuous, we have that

(3.37) \quad \liminf_{m \to \infty} \int_{\mathbb{R}^n} \bar{a}_{\alpha \beta}(\bar{u}_m) D_{\alpha} u_m D_{\beta} u_m \, dx \geq \int_{\mathbb{R}^n} \bar{a}_{\alpha \beta}(u) D_{\alpha} u D_{\beta} u \, dx

On the other hand, by (3.35) we have that

$$
\lim_{m \to \infty} \int_{\mathbb{R}^n} \bar{h}|u_m|^2 \, dx = \int_{\mathbb{R}^n} \bar{h}|u|^2 \, dx
$$

and so we get that

$$
I_\lambda \geq \liminf_{m \to \infty} \int_{\mathbb{R}^n} \bar{a}_{\alpha \beta}(\bar{u}_m) D_{\alpha} u_m D_{\beta} u_m \, dx + \lim_{m \to \infty} \int_{\mathbb{R}^n} \bar{h}|u_m|^2 \, dx
$$

$$
\geq \int_{\mathbb{R}^n} [\bar{a}_{\alpha \beta}(u) D_{\alpha} u D_{\beta} u + \bar{h}|u|^2] \, dx.
$$

But

$$
\int_{\mathbb{R}^n} |u|^p \, dx = \lambda
$$
and so
\[ I^\infty_\lambda = \int_{\mathbb{R}^n} \left[ \tilde{a}_{ab}(u) D_a u^i D_b u^i + \tilde{h}|u|^2 \right] dx \]
so \( I^\infty_\lambda \) is achieved and Theorem 2.1 is proved.

In the case of \( I_\lambda \), by (3.33) we still have (3.35) with \( \bar{u}_m(x) = u_m(x + y_m) \). If there is \( \lambda_0 \in (0, \lambda) \) such that \( I_{\lambda_0} = I^\infty_{\lambda_0} \), then by Theorem 2.1 there exists \( u_0 \in E \) with \( \int_{\mathbb{R}^n} |u_0|^p dx = \lambda_0 \) and such that \( I^\infty_{\lambda_0} = I^\infty(u_0) \), and hence \( I_{\lambda_0} \leq I(u_0) \leq I^\infty(u_0) = I^\infty_{\lambda_0} \) implies that \( I(u_0) = I_{\lambda_0} \) and therefore \( I_{\lambda_0} \) is achieved by \( u_0 \) Theorem 2.2 is proved.

Now we assume that for any \( 0 < \mu \leq \lambda \) we have \( I_\mu \leq I^\infty_\mu \). If \( (y_m) \) is unbounded, say \( |y_m| \to \infty \), we have, by (ii) of Section 2 and (3.35), that
\[ a_{ab}(x + y_m, \bar{u}_m) \to \tilde{a}_{ab}(u) \quad \text{a.e. in } \mathbb{R}^n. \]
So we have, as in (3.37), that
\[ \liminf_{i \to \infty} \int_{\mathbb{R}^n} a_{ab}(x + y_m, \bar{u}_m) D_a u^i_m D_b u^i_m \, dx \geq \int_{\mathbb{R}^n} \tilde{a}_{ab}(u) D_a u^i D_b u^i \, dx. \]

By (v) of Section 2, (3.35) and the Lebesgue’s theorem we have that
\[ \lim_{m \to \infty} \int_{\mathbb{R}^n} h(x + y_m)|\bar{u}_m|^2 \, dx = \int_{\mathbb{R}^n} \tilde{h}|u|^2 \, dx. \]
Combining (3.38), (3.39) and
\[ \int_{\mathbb{R}^n} |u|^p \, dx = \lambda \]
we have that
\[ I_\lambda = \lim_{m \to \infty} \int_{\mathbb{R}^n} \left[ a_{ab}(x, u_m) D_a u^i_m D_b u^i_m + h(x)|u_m|^2 \right] dx \]
\[ = \lim_{m \to \infty} \int_{\mathbb{R}^n} \left[ a_{ab}(x + y_m, \bar{u}_m) D_a \bar{u}_m^i D_b \bar{u}_m^i + h(x + y_m)|\bar{u}_m|^2 \right] dx \]
\[ \geq \int_{\mathbb{R}^n} \left[ \tilde{a}_{ab}(u) D_a u^i D_b u^i + \tilde{h}|u|^2 \right] dx \]
\[ \geq I^\infty_\lambda \]
which contradicts that \( I_\mu < I^\infty_\mu \) for any \( 0 < \mu \leq \lambda \). Thus we have \( |y_m| \leq C \) and by (3.34) we see that for any \( \epsilon > 0 \), there is a \( R(\epsilon) > 0 \) such that
\[ \int_{|x| > R} \left[ |Du_m|^2 + |u_m|^2 \right] dx \leq \epsilon \]
and hence we may assume the existence of a $u_0 \in E$ such that

$$
\begin{align*}
&u_m \to u_0 \quad \text{in } E \\
u_m \to u_0 \quad \text{in } H^1(\mathbb{R}^n), \\
u_m' \to u_0' \quad \text{in } L^2(\mathbb{R}^n) \\
&2 \leq t < \frac{2n}{n-2} \quad (1 \leq i \leq N),

&u_m \to u_0 \quad \text{a.e. in } \mathbb{R}^n \\
&\int_{\mathbb{R}^n} |u_0|^p \, dx = \lambda
\end{align*}
$$

Thus, similarly to (3.38) and (3.39) we can prove that

$$
\lim_{m \to \infty} \int_{\mathbb{R}^n} a_{\alpha\beta}(x, u_m) D_\alpha u_m' D_\beta u_m' \, dx \geq \int_{\mathbb{R}^n} a_{\alpha\beta}(x, u_0) D_\alpha u_0' D_\beta u_0' \, dx
$$

$$
\lim_{m \to \infty} \int_{\mathbb{R}^n} h(x) |u_m|^2 \, dx = \int_{\mathbb{R}^n} h(x) |u_0|^2 \, dx.
$$

Since $\int_{\mathbb{R}^n} |u_0|^p \, dx = \lambda$ we have

$$
I_\lambda \geq \liminf_{m \to \infty} \int_{\mathbb{R}^n} \left[ a_{\alpha\beta}(x, u_m) D_\alpha u_m' D_\beta u_m' + h(x) |u_m|^2 \right] \, dx
$$

$$
\geq \int_{\mathbb{R}^n} \left[ a_{\alpha\beta}(x, u_0) D_\alpha u_0' D_\beta u_0' + h(x) |u_0|^2 \right] \, dx
$$

$$
\geq I_\lambda
$$

and hence $I_\lambda$ is achieved by $u_0 \in E$. Theorem 2.2 is proved.

4. Proof of Theorem 2.3

In this section, we prove Theorem 2.3. The main difficulty is that $I_\lambda$ is in general not in $C^1(E, \mathbb{R})$. To overcome this difficulty, we first prove that

$$
\frac{d}{dt} f \left( \frac{\lambda(u + t_\phi)}{|u + t_\phi|^p} \right) \bigg|_{t_0}
$$

exists for special $\phi \in E$ and then show that $|u|^p$ is finite where $u$ is a minimizer of $I_\lambda$ for some $\lambda > 0$. Finally we prove the theorem.

PROOF OF THEOREM 2.3. By Theorem 2.2 we may assume without loss of generality the existence of $u \in E$, with $\int_{\mathbb{R}^n} |u|^p \, dx = 1$ and such that

$$
I_1 = \int_{\mathbb{R}^n} \left[ a_{\alpha\beta}(x, u) D_\alpha u^i D_\beta u^i + h(x) |u|^2 \right] \, dx.
$$
We first prove that for any $\tau \geq 0$,

\[
\frac{d}{dt} I\left( \frac{u + t|u|^\tau u}{\|u + t|u|^\tau u\|_p} \right) \bigg|_{t=0} = 0
\]

where

\[
|u|_L = \begin{cases} 
|u| & \text{if } |u| \leq L \\
L & \text{if } |u| > L
\end{cases}
\]

It is easy to see that $u + t|u|^\tau u = (1 + t|u|^\tau)u \in E$ for any $t \geq 0$ and since $u$ achieves $I_1$, (4.1) will hold if

\[
\frac{d}{dt} I\left( \frac{u + t|u|^\tau u}{\|u + t|u|^\tau u\|_p} \right) \bigg|_{t=0}
\]

exists.

Because $0 \leq |u|_L^\tau \leq L^\tau$, there is a $M > 0$, depending on $\beta$ and $L$, such that

\[
\frac{1}{2} \leq \|u + t|u|^\tau u\|_p \leq M
\]

for $t$ small enough.

It is easy to prove that

\[
\frac{d}{dt} \left( \|u + t|u|^\tau u\|_p \right) \bigg|_{t=0} = \int_{\mathbb{R}^n} |u|^p |u|^\tau \, dx
\]

and hence

\[
\frac{d}{dt} \left[ \int_{\mathbb{R}^n} h(x)|u + t|u|^\tau u|^2 \frac{dx}{\|u + t|u|^\tau u\|_p} \right] \bigg|_{t=0} = 2 \int_{\mathbb{R}^n} h(x)|u|^2 |u|^\tau \, dx - 2 \int_{\mathbb{R}^n} h(x)|u|^2 \int_{\mathbb{R}^n} |u|^p |u|^\tau \, dx.
\]

On the other hand

\[
I\left( \frac{u + t|u|^\tau u}{\|u + t|u|^\tau u\|_p} \right) = \int_{\mathbb{R}^n} a_{ab}(x - u + t|u|^\tau u - u + t|u|^\tau u) \frac{D_a u^i D_b u^i}{\|u + t|u|^\tau u\|_p^2} \, dx
\]

\[
+ 2t \int_{\mathbb{R}^n} a_{ab}(x - u + t|u|^\tau u - u + t|u|^\tau u) \frac{D_a u^i D_b u^i}{\|u + t|u|^\tau u\|_p^2} \, dx
\]

\[
+ t^2 \int_{\mathbb{R}^n} a_{ab}(x - u + t|u|^\tau u - u + t|u|^\tau u) \frac{D_a (|u|^\tau u^j) D_b (|u|^\tau u^j)}{\|u + t|u|^\tau u\|_p^2} \, dx
\]

\[
+ \int_{\mathbb{R}^n} h(x)|u + t|u|^\tau u|^2 \frac{dx}{\|u + t|u|^\tau u\|_p^2}
\]
\begin{equation}
(4.5) \quad \int \left( u + t|u|^\gamma_L u \right) \frac{\|u + t|u|^\gamma_L u\|^2_p}{\|u + t|u|^\gamma_L u\|_p} \, dx = I^1(t) + I^2(t) + I^3(t) + \int_{\mathbb{R}^n} \frac{h(x)|u + t|u|^\gamma_L u|^2}{\|u + t|u|^\gamma_L u\|^2_p} \, dx
\end{equation}

Using (ii), (iii) of Section 2, (4.2) and (2.1), the inequality

\begin{equation}
\left| a_{\alpha\beta}(x, u + t|u|^\gamma_L u) \frac{u + t|u|^\gamma_L u}{\|u + t|u|^\gamma_L u\|_p} \left| u + t|u|^\gamma_L u \right|^2 \right| \leq C |D_\alpha u| D_\beta |u|^\gamma_L u^\gamma, L^1(\mathbb{R}^n)
\end{equation}

(which holds if \(|u| \leq L\)) and the Dominated Convergence Theorem, we have that

\begin{equation}
(4.6) \quad \frac{d}{dt} I^2(t) \bigg|_{t=0} = \lim_{t \to 0} \frac{I^2(t)}{t} = 2 \int_{\mathbb{R}^n} a_{\alpha\beta}(x, u) D_\alpha u D_\beta D_\beta (|u|^\gamma_L u^\gamma) \, dx.
\end{equation}

Similarly, we have that

\begin{equation}
(4.7) \quad \frac{d}{dt} I^3(t) \bigg|_{t=0} = 0
\end{equation}

On the other hand

\begin{equation}
\frac{d}{dt} I^3(t) \bigg|_{t=0} = \lim_{t \to 0} \int_{\mathbb{R}^n} \frac{1}{t} \left[ a_{\alpha\beta}(x, u + t|u|^\gamma_L u) \left| u + t|u|^\gamma_L u \right|^{-2} \left[ u + t|u|^\gamma_L u \right] D_\alpha u D_\beta u^\gamma \right] \, dx
\end{equation}

\begin{equation}
= \lim_{t \to 0} \int_{\mathbb{R}^n} \frac{1}{t} \left[ a_{\alpha\beta}(x, u + t|u|^\gamma_L u) \left| u + t|u|^\gamma_L u \right|^{-2} D_\alpha u D_\beta u^\gamma \right] \, dx
\end{equation}

\begin{equation}
+ \lim_{t \to 0} \frac{1}{t} \left( \left( \left| u + t|u|^\gamma_L u \right|^{-2} - 1 \right) \int_{\mathbb{R}^n} a_{\alpha\beta}(x, u) D_\alpha u D_\beta u^\gamma \right) \, dx
\end{equation}

\begin{equation}
= \lim_{t \to 0} I^3(t) + \lim_{t \to 0} I^5(t).
\end{equation}

By (4.3), we have

\begin{equation}
(4.8) \quad \lim_{t \to 0} I^4(t) = -2 \int_{\mathbb{R}^n} |u|^\gamma_L u^\gamma \, dx \int_{\mathbb{R}^n} a_{\alpha\beta}(x, u) D_\alpha u D_\beta u^\gamma \, dx.
\end{equation}

Using the mean value theorem we get that

\begin{equation}
\lim_{t \to 0} I^4(t) = \lim_{t \to 0} \int_{\mathbb{R}^n} D_\omega / a_{\alpha\beta}(x, u + t|u|^\gamma_L u) \left[ \frac{|u|^\gamma_L u^\gamma}{\|u + t|u|^\gamma_L u\|^2_p} - \frac{u^\gamma + t|u|^\gamma_L u^\gamma}{\|u + t|u|^\gamma_L u\|^2_p} \frac{d}{dt} \|u + t|u|^\gamma_L u\|_{p(t=t')} \right]
\end{equation}

\begin{equation}
|u + t|u|^\gamma_L u\|^2 D_\alpha u D_\beta u^\gamma \, dx
\end{equation}
\begin{align}
&= \lim_{t \to 0} \int_{\mathbb{R}^n} D_u a_{ab} \left( x, \frac{u + t' |u|^p u}{|u + t' |u|^p|u|^p} \right) \frac{|u|^p u}{|u + t' |u|^p|u|^p} - \int_{\mathbb{R}^n} D_u a_{ab} \left( x, \frac{u + t' |u|^p u}{|u + t' |u|^p|u|^p} \right) \frac{|u|^p u}{|u + t' |u|^p|u|^p} \, dx \\
&- \int_{\mathbb{R}^n} \frac{u^j + t' |u|^p u^j}{|u + t' |u|^p|u|^p} \frac{d}{dt} \left( |u + t |u|^p|u|^p \right) - \int_{\mathbb{R}^n} \frac{D_u u^j D_{\beta} u}{|u + t |u|^p|u|^p} \, dx
\end{align}

(4.9) \quad \lim_{t \to 0} I^p(t) - \lim_{t \to 0} I^p(t) \quad (0 < t' = t'(\lambda) < t)

By (vi) of Section 2 and (3.2) we have that

\begin{align}
&\left| D_u a_{ab} \left( x, \frac{u + t' |u|^p u}{|u + t' |u|^p|u|^p} \right) \frac{|u|^p u}{|u + t' |u|^p|u|^p} - \frac{D_u u^j D_{\beta} u}{|u + t |u|^p|u|^p} \right| \\
&\leq C \left( \frac{|u + t' |u|^p u}{|u + t' |u|^p|u|^p} \right) \frac{|u|^p u}{|u + t |u|^p|u|^p} |Du|^2
\end{align}

hence by the Dominated Convergence Theorem

(4.10) \quad \lim_{t \to 0} I^p(t) = \int_{\mathbb{R}^n} |u|^p |u|^p d\lambda \int_{\mathbb{R}^n} u^j D_{\beta} u \, dx.

Similarly, by (vi) of Section 2, (4.2) and (4.3) we get

(4.11) \quad \lim_{t \to 0} I^p(t) = \int_{\mathbb{R}^n} |u|^p |u|^p d\lambda \int_{\mathbb{R}^n} u^j D_{\beta} u \, dx.

Combining (4.4)-(4.11) we see that (4.1) holds and that

\begin{align}
0 &= \int_{\mathbb{R}^n} a_{ab}(x, u) D_a u^j D_{\beta} (|u|^p u^j) \, dx \\
&+ \int_{\mathbb{R}^n} |u|^p |u|^p d\lambda \int_{\mathbb{R}^n} u^j D_{\beta} u \, dx
\end{align}

Similarly, by (vi) of Section 2, (4.2) and (4.3) we get

(4.11) \quad \lim_{t \to 0} I^p(t) = \int_{\mathbb{R}^n} |u|^p |u|^p d\lambda \int_{\mathbb{R}^n} u^j D_{\beta} u \, dx.

Combining (4.4)-(4.11) we see that (4.1) holds and that

\begin{align}
0 &= \int_{\mathbb{R}^n} a_{ab}(x, u) D_a u^j D_{\beta} (|u|^p u^j) \, dx \\
&+ \int_{\mathbb{R}^n} |u|^p |u|^p d\lambda \int_{\mathbb{R}^n} u^j D_{\beta} u \, dx
\end{align}
which implies that

\[(4.12) \int_{\mathbb{R}^n} a_{\alpha\beta}(x, u) D_\alpha u' D_\beta u' \frac{|u|^2}{L} \, dx + \frac{1}{2} \int_{\mathbb{R}^n} h(x)|u|^2 \, dx = \lambda \int_{\mathbb{R}^n} |u|^p |u|_L^r \, dx \quad \text{(for every } \tau \geq 0 \text{ and } L \geq 0),\]

where

\[\lambda = \int_{\mathbb{R}^n} [a_{\alpha\beta}(x, u) D_\alpha u' D_\beta u' + \frac{1}{2} u' D_\alpha a_{\alpha\beta}(x, u) D_\alpha u' D_\beta u' + h(x)|u|^2] \, dx.\]

Now we are ready to prove that \( |u|_\infty < +\infty \). By (4.12), we have for any \( \tau \geq 0 \), that

\[(4.13) \int_{\mathbb{R}^n} |u|_L^{r-1} u' a_{\alpha\beta}(x, u) D_\alpha u' D_\beta u' \, dx + \tau \int_{\mathbb{R}^n} |u|_L^{r-1} u' a_{\alpha\beta}(x, u) D_\alpha u' D_\beta u' |u|_L \, dx \]
\[+ \frac{1}{2} \int_{\mathbb{R}^n} |u|_L^{r-1} u' D_\alpha a_{\alpha\beta}(x, u) D_\alpha u' D_\beta u' \, dx + \int_{\mathbb{R}^n} h(x)|u|^2 |u|_L^r \, dx = \lambda \int_{\mathbb{R}^n} |u|^p |u|_L^r \, dx.\]

It is easy to see that

\[\int_{\mathbb{R}^n} |u|_L^{r-1} u' a_{\alpha\beta}(x, u) D_\alpha u' D_\beta u' \, dx = \int_{\mathbb{R}^n} |u|_L^{r-1} a_{\alpha\beta}(x, u) D_\alpha |D_\beta u| \, dx = \int_{|u| \leq L} |u|_L^{r-1} a_{\alpha\beta}(x, u) D_\alpha |D_\beta u| \, dx \geq 0.\]

So by (2.8) we have

\[(4.14) \quad (1 - a_3) \int_{\mathbb{R}^n} |u|_L^{r-1} a_{\alpha\beta}(x, u) D_\alpha u' D_\beta u' \, dx \leq \lambda \int_{\mathbb{R}^n} |u|^p |u|_L^r \, dx \]

hence

\[\mu (1 - a_3) \int_{\mathbb{R}^n} |D u|^2 |u|_L^r \, dx \leq \lambda \int_{\mathbb{R}^n} |u|^p |u|_L^r \, dx.\]

It is easy to see that

\[|D| u| \leq |D u|^2\]

and from this and (4.14) we get that

\[\mu (1 - a_3) \int_{\mathbb{R}^n} |D| u| |u|_L^r \, dx \leq \lambda \int_{\mathbb{R}^n} |u|^p |u|_L^r \, dx.\]

Thus, there is a \( C > 0 \) such that for any \( \tau \geq 0 \)

\[(4.15) \quad \int_{\mathbb{R}^n} |D u| |u|_L^r \, dx \leq C \int_{\mathbb{R}^n} |u|^p |u|_L^r \, dx\]

holds.
By (4.15) we have, for any $\tau \geq 1$, that

$$
\int_{\mathbb{R}^n} |D|u| \ |u|_{L^\tau}^{r-1} |^2 \ dx \leq C \int_{\mathbb{R}^n} |u|^p |u|_{L^2}^{2(r-1)} \ dx.
$$

Let $w_L = |u| |u|_{L^\tau}^{r-1}$, then we have

$$
Dw_L = D|u| |u|_{L^\tau}^{r-1} + (\tau - 1)|u|_{L^\tau}^{r-2}D|u|_{L^\tau}^{r-2}D|u| |u|.
$$

Thus

$$
\int_{\mathbb{R}^n} |Dw_L|^2 \ dx \leq C \left[ \int_{\mathbb{R}^n} |D|u| |u|_{L^\tau}^{r-1} |^2 \ dx + (\tau - 1)^2 \int_{\mathbb{R}^n} |u|_{L^\tau}^{r-2} |D|u| |u|_{L^\tau} |^2 \ dx \right]
$$

$$
\leq C \left[ \int_{\mathbb{R}^n} |D|u| |u|_{L^\tau}^{r-1} |^2 \ dx + (\tau - 1)^2 \int_{\left( |u| \leq L \right)} |D|u| |u|_{L^\tau}^{r-1} |^2 \ dx \right]
$$

$$
\leq C(1 + (\tau - 1)^2) \int_{\mathbb{R}^n} |D|u| |u|_{L^\tau}^{r-1} |^2 \ dx
$$

$$
\leq C\tau^2 \int_{\mathbb{R}^n} |u|^p |u|_{L^2}^{2r-2} \ dx.
$$

So we get

$$
\int_{\mathbb{R}^n} |Dw_L|^2 \ dx \leq C\tau^2 \int_{\mathbb{R}^n} |u|^p |u|_{L^\tau}^{r-2} w_L^2 \ dx
$$

By (4.17), the Sobolev embedding theorems and Hölder’s inequality we have that

$$
\left| w_L \right|^p_{L^\frac{2\tau}{\tau - 2}} \leq C \left| D w_L \right|_p \left| w_L \right|_{L^\tau}^{\frac{2\tau}{\tau - 2}}
$$

$$
\leq C\tau^2 \left( \int_{\mathbb{R}^n} |u|_{L^{p-2}}^{2r-2} \ dx \right)^{\frac{2}{r-2}} \left( \int_{\mathbb{R}^n} |w_L|_{L^\frac{2r}{r-2}} \ dx \right)^{\frac{2r}{2r - 2}}
$$

$$
= C\tau^2 \left| u \right|_{L^{\frac{2r}{r-2}}} \left| w_L \right|_{L^\frac{2r}{r-2}}.
$$

where $2q/(q - 2) = 2 \cdot 2^*(2 - (p - 2))$, i.e. $q = 2 \cdot 2^*/(p - 2)$. It is easy to see that $q > n$ when $n > 2$ or $n \leq 2$ by choosing $2^*$ large enough, hence

$$
2^* > 2^* > 2q/(q - 2). \text{ If } |u|^p \in L^{2q/(q - 2)}(\mathbb{R}^n), \text{ letting } L \rightarrow +\infty \text{ in (4.18) and using the Dominated Convergence Theorem and Fatou's lemma together with the fact that } |w_L| \leq |u| \text{ we get that}
$$

$$
\left| \left| u \right| \right|_{L^\frac{2\tau}{\tau - 2}} \leq C\tau^2 \left| u \right|_{L^\frac{2r}{r-2}}.
$$

Thus $u \in L^{2q/(q - 2)}(\mathbb{R}^n)$ implies that $u \in L^{2^*}(\mathbb{R}^n)$. If we set $q^* = 2q/(q - 2)$, $\chi = 2^*/q^*$ then $\tau q^* = \tau 2^*$ and we have that

$$
\left| \left| u \right| \right|_{L^{2^*}} \leq C\tau \left| u \right|_{L^{q^*}}
$$

that is

$$
\left| \left| u \right| \right|_{\tau q^*} \leq C^{1/\tau} \tau^{1/\gamma} \left| u \right|_{q^*}.
$$
Let $\tau = \chi^m$, $m = 0, 1, \ldots$, then we have

\[ (4.19) \quad |u|_{\chi^{\tau_2}^{\ast}} \leq \prod_{m=0}^{N-1} (C\chi^m)^{-\frac{m}{\tau}} |u|_{\chi^{\tau_2}^{\ast}} \leq C^{\tau_2} \chi^{\tau} |u|_{\chi^{\tau_2}^{\ast}} \leq C |u|_{\chi^{\tau_2}^{\ast}} \]

where

\[ \sigma = \sum_{m=0}^{N-1} \chi^{-m}, \quad \tau = \sum_{m=0}^{N-1} m\chi^{-m} \]

and $C$ is independent of $N$ for $\sum_{m=0}^{\infty} \chi^{-m}, \sum_{m=0}^{\infty} m\chi^{-m}$ are all convergent. Letting $N \to \infty$ in (4.19) we get

\[ (4.20) \quad |u|_{\infty} \leq C |u|_{\chi^{\tau_2}^{\ast}} < +\infty. \]

Thus $u \in L_{\infty} \cap E$.

Finally, we show that for any $\varphi \in L_{\infty} \cap E$, we have

\[ (4.21) \quad \frac{d}{dt} I\left( \frac{u + t\varphi}{\|u + t\varphi\|_p} \right) \bigg|_{t=0} = 0. \]

Note that we only need to show that

\[ \frac{d}{dt} I\left( \frac{u + t\varphi}{\|u + t\varphi\|_p} \right) \bigg|_{t=0} \]

exists for any $\varphi \in L_{\infty} \cap E$. (4.21) can be proved by using the same method for proving (4.1). In fact, similarly to (4.2), (4.3), (4.4) and (4.5) we may obtain

\[ (4.22) \quad \frac{1}{2} \leq \|u + t\varphi\|_p \leq M \quad \text{(for $t$ small enough)} \]

\[ (4.23) \quad \frac{d}{dt} \|u + t\varphi\|_p \big|_{t=0} = \int_{\mathbb{R}^n} |u|^{p-2} u\varphi' \, dx \]

\[ (4.24) \quad \frac{1}{2} \int_{\mathbb{R}^n} h(x)|u + t\varphi|^2 \, dx \bigg|_{t=0} = 2 \int_{\mathbb{R}^n} h(x)|u\varphi'| \, dx \]

\[ - 2 \int_{\mathbb{R}^n} h(x)|u|^2 \, dx \int_{\mathbb{R}^n} |u|^{p-2} u\varphi' \, dx \]

\[ (4.25) \quad I\left( \frac{u + t\varphi}{\|u + t\varphi\|_p} \right) = \int_{\mathbb{R}^n} a_{\alpha\beta} \left( x, \frac{u + t\varphi}{\|u + t\varphi\|_p} \right) \frac{D_\alpha u D_\beta u'}{\|u + t\varphi\|_p^2} \, dx \]

\[ + 2t \int_{\mathbb{R}^n} a_{\alpha\beta} \left( x, \frac{u + t\varphi}{\|u + t\varphi\|_p} \right) \frac{D_\alpha u D_\beta u'}{\|u + t\varphi\|_p^2} \, dx \]
\[
\begin{align*}
&+ \frac{t^2}{\|u + t\varphi\|^p_p} \int_{\mathbb{R}^n} a_{\alpha\beta}(x, \frac{u + t\varphi}{\|u + t\varphi\|^p_p}) D_{\alpha\beta} \varphi^i \varphi^j \, dx \\
&+ \int_{\mathbb{R}^n} h(x) \|u + t\varphi\|^2_p \, dx \\
= J^1(t) + J^2(t) + J^3(t) + \int_{\mathbb{R}^n} h(x) \|u + t\varphi\|^2_p \, dx.
\end{align*}
\]

Using that \(|u|_\infty \leq C, \|\varphi\|_\infty \leq C\), (4.22) and (ii) of Section 2, and similarly to (4.6) and (4.7) we obtain that

\[
\left. \frac{d}{dt} J^2(t) \right|_{t=0} = 2 \int_{\mathbb{R}^n} a_{\alpha\beta}(x, u) D_{\alpha} u' D_{\beta} \varphi' \, dx, \quad \left. \frac{d}{dt} J^3(t) \right|_{t=0} = 0.
\]

On the other hand, we have that

\[
\left. \frac{d}{dt} J^1(t) \right|_{t=0} = \lim_{t \to 0} \frac{1}{t} \left[ a_{\alpha\beta}(x, \frac{u + t\varphi}{\|u + t\varphi\|^p_p}) \|u + t\varphi\|^{-2}_p - a_{\alpha\beta}(x, u) \right] D_{\alpha} u' D_{\beta} u' \, dx
\]

\[
= \lim_{t \to 0} \int_{\mathbb{R}^n} \frac{1}{t} \left[ a_{\alpha\beta}(x, \frac{u + t\varphi}{\|u + t\varphi\|^p_p}) - a_{\alpha\beta}(x, u) \right] \|u + t\varphi\|^{-2}_p D_{\alpha} u' D_{\beta} u' \, dx
\]

\[
+ \lim_{t \to 0} \frac{1}{t} (\|u + t\varphi\|^{-2}_p - 1) \int_{\mathbb{R}^n} a_{\alpha\beta}(x, u) D_{\alpha} u' D_{\beta} u' \, dx
\]

\[
= \lim_{t \to 0} J^4(t) + \lim_{t \to 0} J^3(t).
\]

By (4.23) and similarly to (4.8) we obtain that

\[
\lim_{t \to 0} J^3(t) = -2 \int_{\mathbb{R}^n} |u|^{-p}_p u' \varphi' \, dx \int_{\mathbb{R}^n} a_{\alpha\beta}(x, u) D_{\alpha} u' D_{\beta} u' \, dx.
\]

Using the mean value theorem we have that

\[
\lim_{t \to 0} J^4(t) = \lim_{t \to 0} \int_{\mathbb{R}^n} D_{\nu} a_{\alpha\beta}(x, \frac{u + t\varphi}{\|u + t\varphi\|^p_p})
\]

\[
\left[ \frac{\varphi^j}{\|u + t\varphi\|^p_p} - \frac{u^j + t\varphi^j}{\|u + t\varphi\|^p_p} \right] d \left. \|u + t\varphi\|_p \right|_{t=t'}.
\]

\[
\|u + t\varphi\|^2_p D_{\alpha} u' D_{\beta} u' \, dx
\]
\[
    \begin{align*}
    &= \lim_{{t \to 0}} \int_{\mathbb{R}^n} D_{u/a_{0b}}(x, u + t' \varphi) \frac{\varphi^j}{\|u + t' \varphi\|_p} \frac{\|u + t' \varphi\|_p^{-2} D_{a_{0}} D_{a_{0}} u d u^j}{u + t' \varphi} dx \\
    &= \lim_{{t \to 0}} \int_{\mathbb{R}^n} D_{u/a_{0b}}(x, u + t' \varphi) \frac{u^j + t' \varphi^j}{\|u + t' \varphi\|_p^2} \frac{\|u + t' \varphi\|_p^{-2} D_{a_{0}} u d u^j}{u + t' \varphi} dx \\
    &= \lim_{{t \to 0}} \left( J^6(t) - \lim_{{t \to 0}} J^7(t) \right),
    \end{align*}
\]

where \(0 < t'(x) < t\). By (2.7) and (4.22) we see that

\[
\left| D_{u/a_{0b}}(x, u + t' \varphi) \frac{\varphi^j}{\|u + t' \varphi\|_p} \frac{\|u + t' \varphi\|_p^{-2} D_{a_{0}} u d u^j}{u + t' \varphi} \right| \\
\leq C_{\eta} \left( \frac{|u| + t' |\varphi|}{\|u + t' \varphi\|_p} \right) \|u + t' \varphi\|_p^{-2} |D u|^2 \\
\leq C |D u|^2 \in L^1(\mathbb{R}^n).
\]

So, by the Dominated Convergence Theorem we have that

(4.26) \quad \lim_{{t \to 0}} J^6(t) = \int_{\mathbb{R}^n} \varphi^j D_{u/a_{0b}}(x, u) D_{a_{0}} u^j D_{a_{0}} u^i dx.

Similarly to (4.11), we have that

(4.27) \quad \lim_{{t \to 0}} J^7(t) = \int_{\mathbb{R}^n} \|u\|^{-2} u^j \varphi^j dx \int_{\mathbb{R}^n} u^j D_{u/a_{0b}}(x, u) D_{a_{0}} u^j D_{a_{0}} u^i dx.

Combining (4.24)-(4.27) we have that

\[
0 = 2 \int_{\mathbb{R}^n} a_{0b}(x, u) D_{a_{0}} u^j D_{a_{0}} u^i dx + \int_{\mathbb{R}^n} \varphi^j D_{u/a_{0b}}(x, u) D_{a_{0}} u^j D_{a_{0}} u^i dx \\
- \int_{\mathbb{R}^n} |u|^{-2} u^j \varphi^j dx \int_{\mathbb{R}^n} u^j D_{u/a_{0b}}(x, u) D_{a_{0}} u^j D_{a_{0}} u^i dx \\
- 2 \int_{\mathbb{R}^n} |u|^{-2} u^j \varphi^j dx \int_{\mathbb{R}^n} a_{0b}(x, u) D_{a_{0}} u^j D_{a_{0}} u^i dx \\
+ 2 \int_{\mathbb{R}^n} h(x) u^j \varphi^j dx - 2 \int_{\mathbb{R}^n} h(x) |u|^2 dx \int_{\mathbb{R}^n} |u|^{-2} u^j \varphi^j dx
\]

which implies that

(4.28) \quad \int_{\mathbb{R}^n} a_{0b}(x, u) D_{a_{0}} u^j D_{a_{0}} u^i dx + \frac{1}{2} \int_{\mathbb{R}^n} \varphi^j D_{u/a_{0b}}(x, u) D_{a_{0}} u^j D_{a_{0}} u^i dx \\
+ \int_{\mathbb{R}^n} h(x) u^j \varphi^j dx = \lambda \int_{\mathbb{R}^n} |u|^{-2} u^j \varphi^j dx
for every \( \varphi \in L_\infty \cap E \) where

\[
\lambda = \int_{\mathbb{R}^n} \left[ a_{\alpha\beta}(x,u)D_\alpha u^i D_\beta u^i + \frac{1}{2} u^i D_\alpha a_{\alpha\beta}(x,u) D_\beta u^i + h(x)|u|^2 \right] dx
\]

i.e. \( u \) is a weak solution of (1.1) with \( \|u\|_\infty < \infty \) and Theorem 2.3 is completely proved. \( \square \)

References


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