ON THE ABSOLUTE RIESZ SUMMABILITY FACTORS FOR FOURIER SERIES AND CONJUGATE SERIES

by

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1. Definitions and Notations.—Let \( L = L(w) \) be a continuous, differentiable and monotonic increasing \((\nearrow)\) function of \( w \), and let it tend to infinity with \( w \). Suppose that \( \sum_{n=1}^{\infty} a_n \) is a given infinite series then

\[
\sum_{n=1}^{\infty} a_n \quad \text{is summable}
\]

or symbolically

\[
\sum_{n=1}^{\infty} a_n \in \mid R, L_r \mid (r > 0),
\]

if

\[
\int_{-\infty}^{\infty} \frac{L'(w)}{|L(w)|^{r+1}} \left| \sum_{n=1}^{\infty} |L(w) - L(u)|^{-1} L(u) a_n \right| \, dw < \infty,
\]

where \( A \) is a finite positive number (Obrechkoff [4], [5]).

Let \( f(t) \) be a periodic function with period \( 2\pi \) and integrable in the sense of Lebesgue over \( (-\pi, \pi) \). Without any loss of generality, the constant term of the Lebesgue-Fourier series of \( f(t) \) can be taken to be zero so that

\[
\int_{-\pi}^{\pi} f(t) \, dt = 0,
\]
and
\[ f(t) \sim \sum_{n=1}^{\infty} \left( a_n \cos nt + b_n \sin nt \right) = \sum_{n=1}^{\infty} A_n(t). \]

Then the conjugate series of Lebesgue-Fourier series is
\[ \sum_{n=1}^{\infty} \left( b_n \cos nt - a_n \sin nt \right) = \sum_{n=1}^{\infty} B_n(t). \]

We use the following notations:

\[ \Phi(t) = \frac{1}{2} \left[ f(x + t) + f(x - t) \right]. \quad (1.1) \]
\[ \Psi(t) = \frac{1}{2} \left[ f(x + t) - f(x - t) \right]. \quad (1.2) \]
\[ \Phi_1(t) = t^{-1} \int_0^t \Phi(u) \, du. \quad (1.3) \]
\[ \Psi_1(t) = t^{-1} \int_0^t \Psi(u) \, du. \quad (1.4) \]
\[ K(w, t) = \sum_{n \in w} \frac{L(n)}{\alpha(n)} \cos nt. \quad (1.5) \]
\[ \bar{K}(w, t) = \sum_{n \in w} \frac{1}{\alpha(n)} \sin nt. \quad (1.6) \]

2. **Introduction.**—In 1948, Cheng [1] proved the following:

**Theorem A.**—If, for \( 0 < \delta < 1 \),
\[ \Phi_\delta(t) \in BV(0, \pi) \]

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1. \( f(x) \in BV(a, b) \) we mean
\[ \int_a^b |df(x)| < \infty. \]
then
\[ \sum_{n=1}^{\infty} A_n(x) \log (n + 1))^{-\epsilon} \in \mathbb{R}, \epsilon \delta, \]
where \( \epsilon > 0 \).

We observe that above theorem is the particular case of a theorem of Dikshit [2].

Generalising Theorem A for \( \delta = 1 \), Lal [3] has proved the following:

**Theorem B.**—If \( \Phi(t) \in B V (0, \pi) \), then
\[ \sum_{n=1}^{\infty} A_n(x)/(\log (n + 1))^{\epsilon \delta} \in \mathbb{R}, \exp |(\log \omega)\delta|, 1 |, \]
for \( \epsilon > 0 \) and \( d > 1 \).

The following analogue of Theorem B, for allied series of Fourier series, also has been established by Lal [3].

**Theorem C.**—If \( \Psi_1(t) \in B V (0, \pi) \) and \( \Psi_1(t) | t^{-1} \) is integrable in the sense of Lebesgue over \( (0, \pi) \), then
\[ \sum_{n=1}^{\infty} B_n(x)/(\log (n + 1))^{\epsilon \delta} \in \mathbb{R}, \exp |(\log \omega)\delta|, 1 |, \]
\( \epsilon > 0 \) and \( d > 1 \).

Some of the objects of this paper are the following:
1. To generalise Theorem B and Theorem C.
2. To obtain refinements of Theorem A for \( \delta = 1 \) and Theorem B by taking more rapidly increasing type of Riesz means and less effective summability factors.
3. To obtain refinements of Theorem C by dropping the condition (2) of Theorem C and taking more rapidly increasing type of Riesz means.

We, precisely, prove the following theorems.

**Theorem 1.**—Let \( L(w) \) and \( a(w) \) be positive and satisfying, for large \( w \), the following conditions:
\[ |L(w)/a(w)| \not\to \text{ with } w \geq w_0 \quad (2.1) \]
and
\[ \int_{\mathbb{R}} \left| \frac{L'(w)}{L(w)} a(w) \right| \, dw < \infty. \tag{2.2} \]

Then, if
\[ \Phi_1(t) \in B V(0, \pi), \sum_{n=1}^{\infty} A_n(x/a(n)) \in \| R, L(w), 1 \|. \]

**Theorem 2.**—Let \( L(w) \) and \( a(w) \) be positive and satisfying, for large \( w \), (2.1) and (2.2). Then, if
\[ \Psi_1(t) \in B V(0, \pi), \sum_{n=1}^{\infty} B_n(x/a(n)) \in \| R, L(w), 1 \|, \]

whenever
\[ \sum_{n=1}^{\infty} n^{-1} (a(n))^{-1} < \infty. \tag{2.3} \]

3. We shall use the following lemmas in the proof of the theorems.

**Lemma 1.** (Obrechkoff [4], [5]).—If
\[ \Sigma a_n \in \| R, L, r' \mid (r \geq 0), \]
then
\[ \Sigma a_n \in \| R, L, r \mid (r' > r). \]

**Lemma 2.**—If \( L(w) \) and \( a(w) \) be positive and satisfy (2.1) then, uniformly in \( 0 < t < \pi \) and for large \( w \),
\[ \frac{K(w, t)}{\overline{K}(w, t)} = O \left\{ \frac{L(w)}{t a(w)} \right\}. \]

**Proof.**—First we consider \( K(w, t) \). We have, where \([w]\) denotes the integral part of \( w \),
\[ K(w, t) = \sum_{n=-1}^{[w]} \frac{L(n)}{a(n)} \cos nt = \left( \sum_{n=1}^{[w]} + \sum_{[w]+1}^{[w]} \right) \left( \frac{L(n)}{a(n)} \cos nt \right) = P + Q, \]
say. Now, since by (2.1)

$$|L(\alpha)/\alpha(\alpha)| \not\in \text{ with } n \geq [w_0] + 1$$

we have by Abel’s lemma

$$Q = O \left\{ \frac{L(w)}{\alpha(w)} \max_{[w_0] \leq \{w\} \leq w - 1} \left| \sum_{n = \lfloor w \rfloor + 1}^{n-t} \cos nt \right| \right\} = O \{t^{-1} L(w)/\alpha(w)\},$$

uniformly in $0 < t < \pi$. And we observe that $P = O(1)$. Therefore combining the results for $P$ and $Q$, we get

$$K(w, t) = O \{L(w)/\alpha(w)\},$$

The proof for $K(w, t)$ runs parallel to that of $K(w, t)$.

4. Proof of Theorem 1.—Since,

$$A_n(x) = \frac{2}{\pi} \int_0^\pi \Phi(t) \cos nt dt,$$

we have, integrating by parts and using $\Phi_1(\pi) = 0$,

$$A_n(x) = \frac{2}{\pi} \left[ t \Phi_1(t) \cos nt \right]^\pi_0 + \frac{2}{\pi} \int_0^\pi \Phi_1(t) nt \sin nt dt =$$

$$= \frac{2}{\pi} \int_0^\pi \Phi_1(t) nt \sin nt dt = \frac{2}{\pi} \left[ \Phi_1 \left( \frac{\sin nt}{n} - t \cos nt \right) \right]^\pi_0 -$$

$$- \frac{2}{\pi} \int_0^\pi \left( \frac{\sin nt}{n} - t \cos nt \right) d \Phi_1(t) =$$

$$= - \frac{2}{\pi} \int_0^\pi \left( \frac{\sin nt}{n} - t \cos nt \right) d \Phi_1(t).$$

The series

$$\sum_{n=1}^\infty A_n(x)/\alpha(\alpha) \in \mathbb{R}, L(w), 1$$
if
\[
\int_{\omega}^{2\pi} \frac{L'(w)}{|L(w)|^2} \left| \sum_{n \in \mathbb{N}} \frac{L(n)}{a(n)} \int_{0}^{\pi} \left( \frac{\sin nt}{n} - t \cos nt \right) d\Phi_1(t) \right| d\omega,
\]
is convergent.

Since,
\[
\int_{0}^{\pi} \left| d\Phi_1(t) \right| < \infty
\]
by hypothesis of Theorem 1, it is sufficient, for the proof of Theorem 1, to show that
\[
I = \int_{\omega}^{2\pi} \frac{L'(w)}{|L(w)|^2} \left| \sum_{n \in \mathbb{N}} \frac{L(n)}{a(n)} \left( \frac{\sin nt}{n} - t \cos nt \right) \right| d\omega = O(1),
\]
uniformly in $0 < t < \pi$. Now, by (1.5) we have
\[
I \leq t \int_{\omega}^{2\pi} \frac{L'(w)}{|L(w)|^2} \left| K(w, t) \right| d\omega + \int_{\omega}^{2\pi} \frac{L'(w)}{|L(w)|^2} \left| \sum_{n \in \mathbb{N}} \frac{L(n)}{a(n)} \frac{\sin nt}{n} \right| d\omega = I_1 + I_2,
\]
say.

By Lemma 2, we have
\[
I_1 = O\left( \int_{\omega}^{2\pi} \frac{L'(w)}{L(w) a(w)} d\omega \right) = O(1),
\]
by (2.2). And for some suitable integer $[w_n]$ such that $\{L(n)/a(n)\} \not\in$ whenever $n \geq [w_n] + 1$, we write
\[
I_2 \leq \int_{\omega}^{2\pi} \frac{L'(w)}{|L(w)|^2} \left| \sum_{n=0}^{[w_n]} \frac{L(n)}{n a(n)} \frac{\sin nt}{n} \right| d\omega + \int_{\omega}^{2\pi} \frac{L'(w)}{|L(w)|^2} \left| \sum_{n=[w_n]+1}^{[w_n]} \frac{L(n)}{n a(n)} \frac{\sin nt}{n} \right| d\omega = I_{2,1} + I_{2,2},
\]
say.
Now, since
\[
\sum_{n=0}^{[\omega n]} \frac{L(n) \sin nt}{n \mu(n)} = O(1),
\]
we follow that
\[
I_{2.1} = O \left( \int_{\pi}^{\pi} \frac{L'(w)}{L(w) \mu(w)} \left( \max_{[\omega n]+1 \leq r \leq [\omega n]} \left| \sum_{\pi=r}^{[\omega n]} \frac{\sin nt}{n} \right| \right) \, dw \right) = O(1),
\]
and by using Abel's lemma, in view of (2.1), we have
\[
I_{2.2} = O \left( \int_{\pi}^{\pi} \frac{L'(w)}{L(w) \mu(w)} \left( \max_{[\omega n]+1 \leq r \leq [\omega n]} \left| \sum_{\pi=r}^{[\omega n]} \frac{\sin nt}{n} \right| \right) \, dw \right) =
\[
= O \left( \int_{\pi}^{\pi} \frac{|L'(w)|}{L(w) \mu(w)} \, dw \right) \quad \text{(uniformly, } 0 < t < \pi) = O(1)_{\nu}
\]
by (2.2).

This completes the proof of Theorem 1.

5. Proof of Theorem 2.—We have
\[
B_n(x) = \frac{2}{\pi} \int_{0}^{\pi} \Psi_1(t) \sin nt \, dt =
\]
\[
= \frac{2}{\pi} \left[ \Psi_1(t) \sin nt \right]_0^\pi - \frac{2}{\pi} \int_{0}^{\pi} \Psi_1(t) nt \, d(t) =
\]
\[
= -\frac{2}{\pi} \int_{0}^{\pi} \Psi_1(t) nt \, d(t) = \frac{2 \Psi_1(\pi) \cos nt}{n \pi} + \frac{2 \Psi_1(\pi) \cos nt}{n \pi} +
\]
\[
+ \frac{2}{\pi} \int_{0}^{\pi} \left( t \sin nt + \frac{\cos nt}{n} \right) \, dt.
\]

Integrating by parts.
The series
\[ \sum_{n=1}^{\infty} B \frac{x}{\alpha(n)} \in R, L(w), 1, \]

if
\[ \int_{0}^{\pi} \frac{L'(w)}{|L(w)|^2} \left| \sum_{n \leq w} \frac{L(n)}{\alpha(n)} B(x) \right| dw \]

is convergent.

Since, by the hypothesis of the theorem, \( \Psi_1(0) \) and \( \Psi_1(\pi) \) are finite and
\[ \int_{0}^{\pi} |d\Psi_1(t)| < \infty, \]

therefore to prove the theorem it is sufficient to show that
\[ I_1 = t \int_{0}^{\pi} \frac{L'(w)}{|L(w)|^2} |K(w, r)| dw = O(1), \]

uniformly in \( 0 < t < \pi \) and
\[ I_2 = \int_{0}^{\pi} \frac{L'(w)}{|L(w)|^2} \left| \sum_{n \leq w} \frac{L(n)}{n \alpha(n)} \right| dw < \infty. \]

The boundedness of \( I_1 \), uniformly in \( 0 < t < \pi \), runs parallel to that of the boundedness of \( I_1 \) of Theorem 1. And the boundedness of \( I_2 \) follows by using Lemma 1 since, by (2.3),
\[ \sum_{n=1}^{\infty} n^{-1} (\alpha(n))^{-1} < \infty. \]

Thus the Proof of Theorem 2 is complete.

-6. We give some of the corollaries of Theorem 1.
COROLLARY 1.—If \( \Phi_1 (t) \in B_1 (0, \pi) \), then

\[
\sum_{n=1}^{\infty} a_n (x) (\log (n + 1))^c \in \mathbb{R}, \exp \left\{ (\log x)^d (\log \log x)^{-1} \right\}, 1 \ |
\]

where \( c > 0 \) and \( h > 1 \).

PROOF.—Let, for \( c > 0 \) and \( h > 1 \),

\[
a (w) = (\log (w + 1))^c \quad \text{and} \quad L (w) = \exp \left\{ (\log w)^d (\log \log w)^{-1} \right\}
\]

in Theorem 1. Then we observe that the conditions (2.1) and (2.2) are satisfied and therefore we follow the proof.

REMARK 1.—We observe that Corollary 1 is the improvement over the particular case, \( s = 1 \), of Cheng’s result (Theorem A).

We also prove the following:

COROLLARY 2.—If \( \Phi_1 (t) \in B_1 (0, \pi) \), then

\[
\sum_{n=1}^{\infty} a_n (x) (\log (n + 1) \log \log (n + 2))^{-d} \in \mathbb{R}, \exp \left\{ (\log w)^d \right\}, 1 \ |
\]

for \( d > 1 \).

PROOF.—Its proof follows from Theorem 1 by taking

\[
a (w) = (\log (w + 1) \log (w + 2))^d
\]

and

\[
L (w) = \exp \left\{ (\log w)^d \right\} (d > 1).
\]

REMARK 2.—It is easy to observe that we replace the absolute summability factor

\[
(\log (n + 1))^{-d-\epsilon} \quad (\epsilon > 0, \ d > 1).
\]

of Theorem B by

\[
(\log (n + 1))^{-d} (\log \log (n + 2))^{-d} (d > 1)
\]

in Corollary 2.
7. Finally we give two corollaries of Theorem 2 which improve Theorem C by dropping the condition (2) of Theorem C and by taking less effective absolute summability factors than Theorem C.

Corollary 3.—If \( \Psi (t) \in BV (0, \pi) \) then, for \( d > 1 < c, \)

\[
\sum_{n=1}^{\infty} B_n(x)/(\log (n+1))^{d'(\log \log (n+2))} \in |K, \exp |(\log \omega)^s|, 1 |.
\]

Corollary 4.—If \( \Psi (t) \in BV (0, \pi) \) then, for \( d + c - 1 > 0 < c, \)

\[
\sum_{n=1}^{\infty} B_n(x)/(\log (n+1))^{d'c} \in |R, \exp |(\log \omega)^s|, 1 |.
\]

References


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