THE JORDAN-HOLDER THEOREM FOR SEMIRINGS

by

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§ 1. Introduction

The object of this paper is to prove the Green and Jordan-Hölder theorems in semirings. We follow Rees [11], Green [5], Clifford and Preston [2]. This work is similar to [7] and generalizes [8] and [9]. Although some proofs are parallels to that for semigroups, we explain them here to obtain a complete and self-contained exposition.

A semiring is a non-empty set $A \neq \emptyset$ together with two associative operations on $A$, named «addition» (+) and «multiplication» ($\cdot$) (or juxtaposition), so that multiplication is left and right distributive with respect to addition. As is customary, if $B, C$ are subsets of $A$ and $d$ is an element of $A$, we write

$$B + C = \{ b + c : b \in B, c \in G \}, \quad B \cdot C = B \cdot C =$$

$$= \{ b \cdot c : b \in B, c \in C \}, \quad B + d = B +$$

$$+ \{ d \}, \quad B \cdot d = B \cdot d = B \cdot \{ d \}.$$

A subset $D$ of $A$ is called subsemiring of $A$ if it verifies $D + D \subseteq D$ and $D \cdot D \subseteq D$. We say that $a \subseteq A$ is a bi-ideal of the semiring $A$ if $a$ satisfies

$$(1) \quad a + A \subseteq a, \quad A + a \subseteq a,$$

and

$$(2) \quad a \cdot A \subseteq a, \quad A \cdot a \subseteq a.$$
We call a bi-ideal of $A$ because is ideal of the semigroups $(A, +)$ and $(A, \cdot)$. We would remark that the word "bi-ideal" was used in Algebra with other meanings [6], [12]. We consider the empty set $\emptyset$ as a bi-ideal and as a subsemiring, called trivial, of every semiring.

An element $w \in A$ will be called distinguished element of the semiring $A$ if it satisfies $w + x = w + y$, $x + w = y + w$ and $x \cdot w = x \cdot y$, for all $x \in A$. A semiring cannot contain two different distinguished elements. We shall always call $\{w\}$ the distinguished element of every semiring, if it exists. A semiring with distinguished element $w$ will be named $w$-semiring and $W$ will always signify the set $\{w\}$.

A bi-ideal $b \neq \emptyset$ of the semiring $A$ generates the following binary relation on $A$:

$$x \mathrel{R_b} y \iff (x = y) \text{ or } (x \in b \text{ and } y \in b),$$

which is an equivalence compatible with the two operations of $A$, that is, $R_b$ is a congruence on $A$. We define $A/b$, called the Rees-factor semiring of $A$ modulus $b$, in this way

$$A / b = A / R_b = (A - b) \cup \{w\},$$

"$-$" meaning the complement set and $\{w\}$ the distinguished element of $A/R_b$.

The reader will verify:

**Proposition 1.**—i) The intersection of any family of bi-ideals of a semiring $A$ is a bi-ideal of $A$.

ii) The set-theoretical union of any family of bi-ideals of $A$ is also a bi-ideal of $A$.

iii) If $m_i$, $i = 1, \ldots, p$ are non-trivial bi-ideals of $A$, then

$$m_1 + m_2 + \ldots + m_p$$

is a non-trivial bi-ideal of $A$ so that

$$m_1 + m_2 + \ldots + m_p \subseteq m_1 \cap m_2 \cap \ldots \cap m_p.$$
§ 2. Simplicity of semirings and minimality of bi-ideals

Definitions.—Improper bi-ideals of the semiring $A$ are $A$, $W$ (if $W$ exists in $A$) and $\emptyset$. A proper bi-ideal of $A$ is every bi-ideal of $A$ that is not improper. We say that a $w$-semiring $A$ is $w$-symmetric (or symmetric) if

$$(3) \quad A + W,$$

and

$$(4) \quad (A + A = W) \quad \text{or} \quad (A \cdot A = W).$$

A primitive symmetric semiring will be a $w$-symmetric semiring without proper bi-ideals. A semiring $A$ will be called $R$-simple ($R$-, from Rees) or simple if and only if $A$ is not $w$-symmetric and $A$ contains no proper bi-ideal. For any $w$-semiring $A$, $k(A) = W$, and we can introduce the concept of $w$-minimal bi-ideal, as usual, in this way: the bi-ideal $m$ of the $w$-semiring $A$ is $w$-minimal if and only if $W$ is the only non-empty bi-ideal of $A$ strictly contained in $m$.

Proposition 2.—Every $w$-minimal bi-ideal $m$ of $A$ constitutes a symmetric or simple semiring.

Proof.—By proposition 1) iii) $(m + m)$ is a bi-ideal of $A$ contained in $m$. Since $m$ is $w$-minimal,

$$m + m = W \quad \text{or} \quad m + m = m.$$ 

In the first case $m$ is symmetric. In the second case, if also $m \cdot m = W$, then $m$ is symmetric and the proposition is proved.

Let $m$ be a $w$-minimal bi-ideal of $A$ so that

$$m + m \pm W, \quad m \cdot m \pm W.$$

We will show, arriving at a contradiction, that $m$ is simple. We have obtained

$$(5) \quad m + m = m.$$
Suppose \( u \) is a proper bi-ideal of \( m \). Then
\[
t = m + m \cup m + m, \ u \not\in u
\]
will be a bi-ideal of \( A \) contained in \( u \), for all \( u \in u \); therefore
\[
(6) \quad m + m \cup m + m = W, \ \forall u \in u.
\]
Consequence of (5), (6) and definitions is
\[
(7) \quad m \cup m = W, \ \forall u \in u.
\]
The set
\[
V \left( m \right) = \left\{ y \in m / m \ y \ m = W \right\}
\]
forms a bi-ideal of \( A \) so that
\[
u \subset V \left( m \right) \subset m;
\]
by the \( u \)-minimality of \( m \):
\[
V \left( m \right) = m;
\]
that is to say
\[
(8) \quad m \cdot m \cdot m = W
\]
Consequence of (8) and hypothesis is
\[
W \subset m \cdot m \subset m
\]
strictly. From this and (5) follows
\[
W \subset m \cdot m = (m + m) \cup (m + m) \subset m \ m + m \ m + m \ m + + m \ m \subset m + m + m
\]
But

\(( m + m m + m )\)

forms a bi-ideal of \( A \) contained in \( m \). Then

\[ m + m \cdot m + m = m. \]

In virtue of this equality together with (8), we deduce

\[ m \cdot m = ( m + m m + m ) m \subseteq m m + m m m + \]
\[ + m m \subseteq m + W + m = W \]

and so we arrive at the contradiction \( m \cdot m = W \).

In another place [10] we show that if \( m \) is a \( w \)-minimal bi-ideal of \( A \) which satisfies

\[ m + m = m, \ m \cdot m = W, \]

then it is possible that \( m \) has proper bi-ideals.

**Proposition 3.**—The bi-ideal kernel \( k \{ A \} \) of a semiring \( A \) (if it exists) is a simple semiring.

In the case that \( A \) is a \( w \)-semiring, the \( k \{ A \} = W \) is simple. We suppose that \( A \) has not distinguished element and that \( u \) is a proper bi-ideal of \( k \{ A \} \). Then

\[ t = k \{ A \} + k \{ A \} u k \{ A \} + k \{ A \} \]

forms a bi-ideal of \( A \) contained in \( u \), a contradiction.

**Proposition 4.**—Let \( A \) be a \( w \)-semiring which only contains as non-empty bi-ideals \( A \) and \( W \); then \( A \) is primitive symmetric or simple semiring.

This proposition is evident.
§ 3. Isomorphism Theorems

We make the convention $A/\emptyset = A$ for any semiring $A$. Also we establish $T/T = W$ for any bi-ideal or subsemiring $T$ of $A$.

**Proposition 5.**—Let $k$ and $P$ be a bi-ideal and a subsemiring of the semiring $A$, respectively; then

i) $k \cup P$ is a subsemiring of $A$ and $k$ forms a bi-ideal of $k \cup P$.

ii) $k \cap P$ is a bi-ideal of the subsemiring $P$.

iii) $\frac{k \cup P}{k} \approx \frac{P}{k \cap P}$

**Proof.**—i) $$(k \cup P) + (k \cup P) \subseteq (k + k) \cup (k + P) \cup (P + k) \cup (P + P) \subseteq k \cup P,$$

$$(k \cup P) \cdot (k \cup P) \subseteq (k \cdot k) \cup (k \cdot P) \cup (P \cdot k) \cup (P \cdot P) \subseteq k \cup P.$$

ii) $$(k \cap P) + P \subseteq (k + P) \cap (P + P) \subseteq k \cap P,$$

$$(k \cap P) \cdot P \subseteq (k \cdot P) \cap (P \cdot P) \subseteq k \cap P.$$

It will be shown similarly

$$P + (k \cap P) \subseteq k \cap P, \quad P \cdot (k \cap P) \subseteq k \cap P.$$

iii) Is true because

$$\frac{k \cup P}{k} = (P - k) \cup \{w\} = \frac{P}{k \cap P}$$

**Proposition 6.**—Let $k \neq \emptyset$ be a bi-ideal of $A$; let $h: A \rightarrow A/k$ be the natural homomorphism. $h$ induces a one-to-one correspondence which preserves inclusion, which we also call $h$

$$h: p \mapsto h(p) = p/k$$
from the set of the bi-ideals of \( A \) that contain \( \mathbf{k} \) upon the set of the non-trivial bi-ideals of \( A/\mathbf{k} \). Moreover,

\[
(A/\mathbf{k})/(\mathbf{p}/\mathbf{k}) = A/\mathbf{p}.
\]

The proof in detail of the above statement does not offer difficulty. It can be made in these steps: i) If \( \mathbf{p} \) is a bi-ideal of \( A \) so that \( \mathbf{k} \subseteq \mathbf{p} \), then \( h(\mathbf{p}) = \mathbf{p}/\mathbf{k} \) is a non-trivial bi-ideal of \( h(A) = A/\mathbf{k} \). ii) If \( \mathbf{q} \) is a non-trivial bi-ideal of \( A/\mathbf{k} \), then \( h^{-1}(\mathbf{q}) = \mathbf{p} \) is a bi-ideal of \( A \) which contains \( \mathbf{k} \), so that \( h(\mathbf{p}) = \mathbf{q} \). Therefore \( h \) induces a mapping from the first set of the statement onto the second. iii) \( h \) induces a one-to-one map from the first set onto the second set, because \( h(\mathbf{m}) = h(\mathbf{n}) \) implies \( \mathbf{m} - \mathbf{k} = \mathbf{n} - \mathbf{k} \) and, therefore, \( \mathbf{m} = \mathbf{n} \). iv) That \( h \) preserves the inclusions is proved in the same manner as iii). v) Lastly,

\[
(A/\mathbf{k})/(\mathbf{p}/\mathbf{k}) = (A/\mathbf{k} - \mathbf{p}/\mathbf{k}) \cup \{w\} = (A - \mathbf{p}) \cup \{w\} = A/\mathbf{p}
\]

**Definitions.**—Given two bi-ideals \( \mathbf{p, q} \) of the semiring \( A \), we say that \( \mathbf{q} \) is *maximal* in \( \mathbf{p} \) if and only if \( \mathbf{q} \subseteq \mathbf{p} \) and there are not any bi-ideal of \( A \) strictly between \( \mathbf{q} \) and \( \mathbf{p} \). We will say that \( \mathbf{m} \) is a *maximal bi-ideal* of \( A \) if \( \mathbf{m} \) is a maximal bi-ideal in \( A \).

**Proposition 7.**—i) Given two bi-ideals \( \mathbf{p} \) and \( \mathbf{q} \neq \emptyset \) of \( A \); \( \mathbf{q} \) is maximal in \( \mathbf{p} \) if and only if \( \mathbf{p}/\mathbf{q} \) forms a \( \mathbf{w} \)-minimal bi-ideal of \( A/\mathbf{q} \). In such a case \( \mathbf{p}/\mathbf{q} \) constitutes a symmetric or simple semiring.

ii) Let \( \mathbf{m} \neq \emptyset \) be-ideals \( A; \mathbf{m} \) is maximal of \( A \) if and only if \( A/\mathbf{m} \neq W \) contains no proper bi-ideal; that is, if and only if \( A/\mathbf{m} \) is primitive symmetric or simple semiring \( \neq W \).

iii) \( \emptyset \) is maximal bi-ideal in \( \mathbf{p} \) if and only if \( \mathbf{k}(A) \) exists and \( \mathbf{p} = \mathbf{k}(A) \). In such a case \( \mathbf{p} \) is simple semiring.

**Proof.**—i) Is a result of proposition 6 and 2. ii) Follows from propositions 6 and 4. iii) Is consequence of definitions and proposition 3.

§ 4. **Principal factors of a semiring**

In this paragraph we extend to semirings part of the theory of J. A. Green about semigroups. See [2] vol. I, chapter 2, section 2.6 and [5].
Let $A$ be a semiring and $x \in A$; $J(x)$ will be the minimal (minimum) bi-ideal of $A$ which contains the element $x$, that is, the intersection of all the bi-ideals of $A$ that contain $x$. We establish an equivalent relation on $A$, which we call $J$ (Green’s relation) in this manner:

$$x, y \in A, x J y \iff J(x) = J(y)$$

Let

$$J_a = \{ u \in A \mid u J_a \}$$

for $a \in A$. Evidently $J_a \subseteq J(a)$, for all $a \in A$. Let $I(a)$ be the set

$$I(a) = J(a) - J_a.$$

**Proposition 8.**

i) $I(a) = \{ t \in A \mid J(t) \subseteq J(a), \text{ strictly} \}$

and ii) $I(a)$ is a bi-ideal of $A$, maximal in $J(a)$.

**Proof.**—i) is evident.

ii) First we prove that $I(a)$ is a bi-ideal of $A$. If $I(a) = \emptyset$, then it is bi-ideal by convention. Assume $I(a) \neq \emptyset$. We show

(9)  

$$I(a) + A \subseteq I(a).$$

Let

$$p \in I(a), x \in A;$$

as $p \in J(p)$, it follows

$$p + x \in J(p);$$

then

$$J(p + x) \subseteq J(p) \subseteq J(a),$$
thanks to part i) of this proposition 8. For the same reason,
\( p + x \in I(a) \), and (9) is proved. Similarly
\[
A + I(a) \subset I(a)
\]

Now we state
\[
(10) \quad I(a) \cdot A \subset I(a).
\]

Let
\[
p \in I(a), x \in A;
\]
then \( p \cdot x \in J(p) \); consequently
\[
J(p \cdot x) \subset J(p) \subset J(a).
\]

Thanks to part i), \( p \cdot x \in I(a) \). Similarly
\[
A \cdot I(a) \subset I(a),
\]
and \( I(a) \) is a bi-ideal of \( A \).

We now deduce that \( I(a) \) is maximal in \( J(a) \). We suppose that \( t \) is a bi-ideal of \( A \) such that
\[
I(a) \subset t \subset J(a)
\]
and let
\[
td t - I(a);
\]
then
\[
J(t) \subset t \subset J(a);
\]
but from
\[
t \in J(a) - I(a) = J_0
\]
we see that
\[
J(t) = t = J(a).
\]
Definitions.—Every Rees factor of the form $J(x)/I_c(x)$, being $x$ an element of the semiring $A$, will be called a principal factor of $A$. All principal factors of $A$ are symmetric or simple semirings, according to propositions 8 and 7. A strictly decreasing chain of bi-ideals of $A$:

$$S_1 = A \supset S_2 \supset \ldots \supset S_p \supset S_{p+1} = \emptyset,$$

which begins with $A$, ends with $\emptyset$ and is such that $S_{i+1}$ is maximal in $S_i$ for

$$i = 1, 2, \ldots, p,$$

will be named a principal series of $A$. Thus $S_p = k(A)$. The factors of the principal series (11), defined by

$$S_i/S_{i+1}, \quad i = 1, \ldots, p,$$

are symmetric or simple semirings by proposition 7. It will be said that two principal series of $A$ are isomorphic if and only if there is a one-to-one correspondence from the set of factors of one onto the set of factors of the other, so that the corresponding factors are isomorphic.

Theorem 1.—Let $A$ be a semiring which admits a principal series. The factors of any principal series of $A$ are isomorphic—taken in a certain order—to the principal factors of $A$. In particular, any two principal series of $A$ are isomorphic. The principal factors of $A$ are symmetric or simple semirings.

Proof.—We consider a factor

$$S_i/S_{i+1}, \quad i = 1, \ldots, p$$

of the principal series (11) of $A$. Let

$$m \in S_i - S_{i+1}; \quad J(m) \cup S_{i+1}$$

is a bi-ideal of $A$ (proposition 1 ii)), which (because it contains $m$ and $S_{i+1}$) contains strictly $S_{i+1}$ and is contained in $S_i$; therefore

$$J(m) \cup S_{i+1} = S_i.$$
We shall see that
\[ I(m) \subseteq S_{i+1}. \]  
(13)

Let
\[ p \notin I(m) \subseteq S_i; \quad \text{if} \quad p \notin S_i \subseteq S_{i+1}, \]
we shall arrive at
\[ J(p) \cup S_{i+1} = S_i, \]
reasoning as before; from which
\[ m \notin J(m), \quad p \notin I(m) \subseteq J(m); \]
\[ \text{it follows } J(p) = J(m); \quad \text{which would imply} \]
\[ p \notin J(m) = J_m = I(m), \]
\[ \text{a contradiction; thus (13) is true,} \]
(14)
\[ I(m) = J(m) \cap S_{i+1}, \quad m \notin S_i \subseteq S_{i+1}, \]
is also true. In effect,
\[ I(m) \subseteq J(m) \cap S_{i+1} \]

because \( I(m) \subseteq J(m) \) and by (13). Let
\[ c \notin J(m) \cap S_{i+1}; \]
then
\[ J(c) \subseteq J(m), \quad J(c) \subseteq S_{i+1} \]
and so \( m \notin J(c); \) from which \( J(c) \subseteq J(m) \) strictly; which, according to proposition 8 i), implies \( c \notin I(m). \) So we have proved (14).
We now use proposition 5, substituting \( \mathbf{k} \) and \( \mathcal{P} \) by \( S_{i+1} \) and \( J(m) \), respectively. From formulas (12) and (14) we see that

\[
S_i / S_{i+1} \cong J(m) / I(m), \quad \forall m \in S_i - S_{i+1}.
\]

The principal factor \( J(m)/I(m) \) is independent of the chosen element \( m \in S_i - S_{i+1} \), as the above proof has shown. Thus the correspondence

\[
\alpha : S_i / S_{i+1} \longrightarrow J(m) / I(m), \quad m \in S_i - S_{i+1}
\]

is a map. \( \alpha \) is one-to-one because, given an arbitrary \( x \in A \), there exists a unique \( i = 1, 2, \ldots, \rho \) so that

\[
x \in S_i - S_{i+1}
\]

\( \alpha \) is a map onto since \( \{ J_x \}_{x \in A} \) is a partition of \( A \) and

\[
J_x = J(x) - I(x) = S_i - S_{i+1},
\]

for

\[
x \in S_i - S_{i+1}.
\]

So the first part of theorem 1 is proved; the other parts are now evident.

§ 5. The Jordan-Hölder Theorem


Proposition 9 (Zassenhaus Lemma).—Let \( R, S \) be two subsemirings of the semiring \( A \); let \( r, s \) be bi-ideals of \( R \) and \( S \), respectively. We call

\[
T = r \cup (R \cap S), \quad t = r \cup (R \cap s),
\]

\[
U = s \cup (R \cap S), \quad u = s \cup (r \cap S).
\]
Then i) $T$ and $U$ are subsemirings of $A$; ii) $t$ and $u$ are bi-ideals of $T$ and $U$, respectively, and iii) 

$$T/t \simeq U/u$$

**Proof.**—Making in proposition 5 i)

$$A = R, \ k = r, \ P = R \cap S,$$

we find that 

$$r \cup (R \cap S) = T$$

is a subsemiring of $R$ and therefore of $A$. We will now show that $t$ is a bi-ideal of $T$. From the fact that $r$ is a bi-ideal of $R$ and $T \subseteq R$ there follows

\begin{equation}
(15)
\begin{aligned}
r + T & \subseteq r, \ T + r \subseteq r.
\end{aligned}
\end{equation}

Taking in proposition 5 ii)

$$A = S, \ k = s, \ P = R \cap S$$

we obtain that

$$s \cap (R \cap S) = R \cap s$$

is a bi-ideal of $R \cap S$. According to this it verifies:

\begin{align}
(16) \quad & (R \cap s) + (R \cap S) \subseteq R \cap s, \quad (R \cap S) + (R \cap s) \subseteq R \cap s, \\
(17) \quad & (R \cap s) \cdot (R \cap S) \subseteq R \cap s, \quad (R \cap S) \cdot (R \cap s) \subseteq R \cap s.
\end{align}

Just as $r$ is a bi-ideal of $R \supseteq R \cap S$ and from (16) we deduce

\begin{align}
(18) \quad & (R \cap s) + T = (R \cap s) + (r \cup (R \cap S)) \subseteq \\
& \subseteq [(R \cap s) + r] \cup [(R \cap s) + (R \cap S)] \subseteq \\
& \subseteq r \cup (R \cap s) = t.
\end{align}
From (15) and (18) it follows that
\[ t + T = [r \cup (R \cap s)] + T \subseteq (r + T) \cup [(R \cap s) + T] \subseteq r \cup t = t \]

Because \( r \) is a bi-ideal of \( R \) and according to (17), we have
\[ t \cdot T = [r \cup (R \cap s)] \cdot [r \cup (R \cap s)] \subseteq \]
\[ \subseteq (r \cdot r) \cup [r \cdot (R \cap S)] \cup [(R \cap s) \cdot r] \cup [(R \cap s) \cdot (R \cap S)] \subseteq \]
\[ \subseteq r \cup (R \cap s) = t \]

Analogically, we can prove
\[ T + t \subseteq t, \quad T \cdot t \subseteq t, \]

Lastly we shall show
\[ T / t \cong U / u. \]

Taking in proposition 5,
\[ A = T, \quad k = t \quad \text{and} \quad P = (R \cap S), \]

we can state
\[ [t \cup (R \cap S)] / t \cong (R \cap S) / [t \cap (R \cap S)]. \]

But
\[ t \cap (R \cap S) = [r \cup (R \cap s)] \cap (R \cap S) \subseteq (r \cap S) \cup (R \cap s). \]

Thus, we arrive at
\[ T / t \cong (R \cap S) / [(r \cap S) \cup (R \cap s)]. \]

We can obtain \( U \) and \( u \) from \( T \) and \( t \) by interchanging \( R, r \) with \( S, s \); the reasoning that led to the conclusion \( t \) is a bi-ideal of \( T \), now leads to the affirmation \( u \) is a bi-ideal of \( U \) and to the isomorphism
\[ U / u \cong (S \cap R) / [(s \cap R) \cup (S \cap r)]. \]
Consequently,

\[ T / t \cong U / u. \]

**Definitions.**—A *series* of a semiring \( A \) is a decreasing chain of sub-semirings of \( A \):

\[ A_1 = A \supset A_2 \supset \ldots \supset A_{r-1} \supset A_r = \emptyset \]

which begins with \( A_1 = A \), ends with \( A_r = \emptyset \), so that \( A_{i+1} \) is a b-ideal of \( A_i \), for \( i = 1, 2, \ldots, r - 1 \). A *refinement* of series (19) is another series of \( A \) which contains, at least, all the terms that are in (19). A *proper refinement* of (19) is a refinement which contains, at least, one term more than the original series (19), different from all those of (19). Two series of \( A \) are called *isomorphic* if there is a one-to-one correspondence from the set of factors of one onto the set of factors of the other, so that the corresponding factors are isomorphic.

**Theorem 2** (The Schreier refinement theorem).—*Any two series of a semiring have isomorphic refinements.*

**Proof.**—Let (19) and

\[ B_1 = A \supset B_2 \supset \ldots \supset B_{r-1} \supset B_r = \emptyset \]

be two series of \( A \). We form the following refinements of (19) and (20):

\[ A_{11} = A \supset A_{12} \supset \ldots \supset A_{1r} = A_{11} \supset \ldots \supset A_{1r} = \emptyset \]

\[ B_{11} = A \supset B_{21} \supset \ldots \supset B_{r1} = B_{11} \supset \ldots \supset B_{r1} = \emptyset \]

where

\[ A_{i,j} = A_{i+1} \cup (A_i \cap B_j) \]

and

\[ B_{i,j} = B_{j+1} \cup (A_i \cap B_j). \]

Proposition 9 asserts that \( A_{i,j}, B_{i,j} \) are subsemirings of \( A \), that.

\[ A_{i,j+1} \quad \text{and} \quad B_{i+1,j} \]
are bi-ideals of $A_{i,j}$ and $B_{i,j}$ respectively, and that

$$A_{i,j} / A_{i,j+1} \cong B_{i,j} / B_{i+1,j}.$$ 

**Definition.**—Every series of the semiring $A$ strictly decreasing which admits no proper refinement will be called a *composition series* of $A$. If (19) is a composition series of $A$ then $A_{i,1}$ is a maximal bi-ideal of $A_i$, for $i = 1, 2, \ldots, r - 1$. In view of proposition 7 and theorem 2, we can state:

**Corollary of theorem 2 (Jordan-Hölder theorem).**—Any two composition series of a semiring are isomorphic (if they exist). The factors of any composition series are primitive symmetric or simple semirings.

The Jordan-Hölder theorem for semirings is not a particular case of the theorem of the same name for universal algebras (see, for example [1] or [3]) because the congruences on a semiring do not commute in general. Neither is it a particular case of the J-H theorem in categories such as is established in [4], as can be easily verified.

In a future work we shall generalize certain results of this paper.

**Bibliography**


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