SOME CHARACTERIZATIONS OF c-PARACOMPACT AND c-COLLECTIONWISE NORMAL SPACES BY CONTINUOUS SELECTIONS

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ABSTRACT

In this note we characterize the c-paracompact and c-collectionwise normal spaces in terms of continuous selections. We include the usual techniques with the required modifications by the cardinality.

1. Introduction

The first results in the theory of continuous selections are due to H. Tong (1948) [24] and C. H. Dowker (1951) [4]. The motivations of these papers were to obtain some characterizations of the normal spaces (H. Tong) and of the countably paracompact and normal spaces (C. H. Dowker).

The systematic development of the selection theory is due to E. Michael [18]-[21]. In [14], Michael shows the characterization of the paracompact and regular, countably paracompact and normal, collectionwise normal and normal spaces by continuous selections.

The c-paracompact spaces were introduced by K. Morita (1962) in [22] obtaining the characterization of such spaces by continuous partitions of the unity.

The c-collectionwise normal spaces were introduced by R. A. Aló and H. L. Shapiro in [6].

In this note we characterize the c-paracompact and c-collectionwise normal spaces in terms of continuous selections. We include the usual techniques with the required modifications by the cardinality.
Given a topological space \( Y \) let \( 2^Y \) denote the set of all non-empty subsets of \( Y \). If \( X, Y \) are topological spaces and \( \phi \) is a mapping of \( X \) into \( 2^Y \) then we say that \( \phi \) is a **multiform mapping or carrier** of \( X \) into \( Y \). If \( \phi \) is a multiform mapping of \( X \) into \( Y \) and \( f \) is a continuous mapping of \( X \) into \( Y \), then \( f \) is called a **selection of** \( \phi \) if \( f(x) \in \phi(x) \) for all \( x \in X \). If \( \phi \) is a multiform mapping from \( X \) into \( Y \), \( \phi \) is called **lower-semicontinuous** if for each open subset \( V \) of \( Y \), it holds that

\[
\left\{ x \in X \mid \phi(x) \cap V \neq \emptyset \right\}
\]

is open in \( X \).

Let \( \phi \) be a multiform mapping of \( X \) into \( Y \). It is clear that \( \phi \) is lower-semicontinuous if and only if:

(A) If \( x \in X \), \( y \in \phi(x) \) and \( V \) is a neighborhood of \( y \), then there exists a neighborhood \( U \) of \( x \) such that for every

\[
x' \in U, \quad \phi(x') \cap V \neq \emptyset.
\]

Let \( (Y, \mathcal{U}) \) be a uniform space and \( \mathcal{U}_0 \) a uniformity in \( Y \) such that \( \mathcal{U}_0 \subset \mathcal{U} \) and \( \mathcal{U}_0 \) has a countable base. A subset \( \mathcal{M} \subset 2^Y \) is **equimetrizable by means of** \( \mathcal{U}_0 \) ([7]) if for every \( U \in \mathcal{U} \) there is a \( V_0 \in \mathcal{U}_0 \) such that

\[
(M \times M) \cap V_0 \subset U
\]

for all \( M \in \mathcal{M} \).

Note that if \( \mathcal{U} \) is pseudo-metrizable, then any subset of \( 2^Y \) is equimetrizable by means of \( \mathcal{U} \).

**Lemma 1** ([14]).—Let \( X \) and \( Y \) be topological spaces, \( V \) a symmetric open set in

\[
Y \times Y
\]

\( \phi \) lower-semicontinuous multiform maps. For each \( x \in X \), let

\[
\theta(x) = \phi(x) \cap V \cup \{ \phi(x) \}.
\]

Then, if \( \theta(x) \neq \emptyset \) for all \( x \in X \), it holds that \( \theta \) is lower-semicontinuous.
2. Some characterizations of c-paracompact spaces by means of selections

Let \( \mathfrak{c} \) be an infinite cardinal. A topological space \( X \) is said to be \textit{c-paracompact} if each open cover of \( X \) of cardinal less or equal than \( \mathfrak{c} \), has an open locally finite refinement.

\textbf{Proposition 2 ([22])}.—Let \( X \) be a topological space and \( \mathfrak{c} \) an infinite cardinal. The following are equivalent:

a) \( X \) is c-paracompact and normal.

b) Every open cover of \( X \) of cardinal less or equal than \( \mathfrak{c} \) has a locally finite continuous partition of unity subordinated to it.

\textbf{Lemma 3}.—Let \( X \) be a normal c-paracompact topological space, \( Y \) a real or complex topological vector space with separability degree less or equal than \( \mathfrak{c} \), \( \psi: X \to 2^Y \) lower-semicontinuous with \( \psi(x) \) convex for all \( x \in X \), and \( V \) a convex open neighborhood of \( 0 \) in \( Y \). Then there is a continuous map \( f: X \to Y \) such that

\[ f(x) \in (\psi(x) + V) \]

for every \( x \in X \).

\textbf{Proof}.—There exits

\[ D = \{ y_i \mid i \in I \} \subset Y \]

such that \( \overline{D} = Y \) and \( \text{card}(I) \leq \mathfrak{c} \), since the separability degree of \( Y \) is less or equal than \( \mathfrak{c} \). For each \( i \in I \), let

\[ U_i = \{ x \in X \mid \psi(x) \cap (y_i - V) \neq \emptyset \} \].

Then

\[ U_i = \{ x \in X \mid \psi(x) \cap (y_i - V) \neq \emptyset \} \]

and by the lower-semicontinuity of \( \psi \), \( U_i \) is an open subset of \( X \) for all \( i \in I \).
Let
\[ \mathcal{U} = \{ U_i \mid i \in I \}. \]

\( \mathcal{U} \) is an open cover of \( X \). In fact, given
\[ x \in X, \quad \phi(x) + V \]
is open in \( Y \) and hence there exits an \( i \in I \) such that
\[ y_i \in (\phi(x) + V). \]

Then \( x \in U_i \).

By Proposition 2, \( \text{card} (I) \leq c \) and \( X \) is c-paracompact and normal, there is a locally finite continuous partition of unity \( \{ \rho_i \mid i \in I \} \) subordinated to \( \mathcal{U} \). Let us consider \( f: X \rightarrow Y \) defined by
\[ f(x) = \sum_{i \in I} \rho_i(x) y_i. \]

Then:
1) \( f \) is well-defined and continuous, since \( \{ \supp(\rho_i) \mid i \in I \} \) is locally finite.
2) For each
\[ x \in X, \quad f(x) \in (\phi(x) + V). \]

In fact, let \( i \in I \) such that \( \rho_i(x) \neq 0 \). Then
\[ x \in \supp(\rho_i) \subset U_i \]

and
\[ y_i \in (\phi(x) + V). \]

Hence
\[ f(x) = \sum_{i \in I} \rho_i(x) y_i \in (\phi(x) + V). \]
since $\psi (x) + V$ is convex and

$$\sum_{i \in I} p_i (x) = 1.$$

**Corollary 4 ([14]).—** Let $X$ be a normal paracompact space, $Y$ a real or complex topological vector space, $\psi : X \to 2^Y$ lower-semi-continuous with $\psi (x)$ convex for every $x \in X$, and $V$ a convex open neighborhood of $0$ in $Y$. There exists a continuous map $f : X \to Y$ such that

$$f (x) \in (\psi (x) + V),$$

for all $x \in X$.

**Theorem 5.** Let $X$ be a topological space and $c$ and infinite cardinal. The following are equivalent:

a) $X$ is normal and c-paracompact.

b) For every set $A$ with $\text{card} (A) \leq c$, and every

$$\phi : X \to 2^c (A)$$

lower-semicontinuous such that $\phi (x)$ is closed and convex for all $x \in X$, it holds that $\phi$ admits a selection.

c) If $Y$ is a real or complex Banach space with separability degree $\leq c$, then every $\phi : X \to 2^Y$ lower-semicontinuous such that $\phi (x)$ is closed and convex for all $x \in X$, admits a selection.

d) If $Y$ is a real or complex Frechet space with separability degree $\leq c$, then every $\phi : X \to 2^Y$ lower-semicontinuous such that $\phi (x)$ is closed and convex for all $x \in X$ admits a selection.

e) If $Y$ is a metrizable, locally convex real or complex topological vector space with separability degree $\leq c$, then every $\phi : X \to 2^Y$ lower-semicontinuous such that $\phi (x)$ is convex and complete for all $x \in X$ admits a selection.

f) If $Y$ is a locally convex, real or complex topological vector
space with separability degree $\leq c$, then every $\phi : X \to 2^Y$ lower-semicontinuous such that $\phi(x)$ is convex and complete for all

$$x \in X \text{ and } \{\phi(x) | x \in X\}$$

is equimetrizable by means of a uniformity $\mathcal{U}_s$ in $Y$, admits a selection.

**Proof.**—a) $\implies$ f).

Let $B_s = \{U_n\}_{n \in \mathbb{N}}$ be a base for $\mathcal{U}_s \subseteq \mathcal{U}$, where $\mathcal{U}$ is the uniformity of $Y$. For $n = 1$, let $V_1$ be a convex open symmetric neighborhood of 0 such that

$$U_{v_1} = \{(x, y) \in Y \times Y | x - y \in V_1\} \subseteq U_1.$$  

For $n = 2$, let $V_2$ be a convex open symmetric neighborhood of 0 such that

$$V_2 + V_2 \subseteq V_1 \quad \text{and} \quad U_{v_n} \subseteq U_2.$$  

Suppose that we have obtained $V_1, \ldots, V_n$ convex open symmetric neighborhoods of 0 such that

$$V_2 + V_2 \subseteq V_2, \ldots, V_n + V_n \subseteq V_{n-1}$$

and

$$U_{v_1} \subseteq U_2, \ldots, U_{v_n} \subseteq U_n.$$  

Let $V_{n+1}$ be a convex open symmetric neighborhood of 0 such that

$$V_{n+1} + V_{n+1} \subseteq V_n \quad \text{and} \quad U_{v_{n+1}} \subseteq U_{n+1}.$$  

By induction, we have the family $\{V_n | n \in \mathbb{N}\}$ of convex open symmetric neighborhoods of 0 such that

$$V_{n+1} + V_{n+1} \subseteq V_n \quad \text{and} \quad U_{v_n} \subseteq U_n.$$
for all \( n \in \mathbb{N} \). Now

\[ B_1 = \{ U_{v_n} \mid n \in \mathbb{N} \} \]

is a base for a uniformity, \( \mathcal{U}_1 \), in \( Y \). In fact:

1) For every

\[ n \in \mathbb{N}, \quad \Delta \subset U_{v_n} . \]

2) For every

\[ m, n \in \mathbb{N}, \quad U_{v_n} \cap U_{v_m} \supset U_{v_{n+m}} . \]

3) For every

\[ n \in \mathbb{N}, \quad U_{v_n}^{-1} = U_{v_n} . \]

4) For every

\[ n \in \mathbb{N}, \quad U_{v_{n+1}} \circ U_{v_{n+1}} \subset U_{v_n} . \]

Moreover \( \mathcal{U}_0 \subset \mathcal{U}_1 \subset \mathcal{U} \). So \( \{ \phi(x) \mid x \in X \} \) is equimetrizable by means of \( \mathcal{U}_1 \).

By induction we will show that there is a sequence \( \{ \phi_{n-1} \} \in \mathbb{N} \) of lower-semicontinuous and multiforms maps of \( X \) into \( Y \) with \( \phi_{n-1}(x) \)

convex for all \( n \in \mathbb{N} \) and \( x \in X \), and a sequence \( \{ f_n \} \in \mathbb{N} \) of continuous maps of \( X \) into \( Y \), such that \( \phi_0 = \phi \) and

\[ \phi_n(x) = (f_n(x) + V_n) \cap \phi_{n-1}(x) \]

for all \( x \in X \) and \( n \in \mathbb{N} \).

Let \( n = 1 \). If we apply Lemma 3 to \( \psi = \phi_0 = \phi \) and \( V = V_1 \), then there is a continuous map \( f_1 \) of \( X \) into \( Y \) such that

\[ f_1(x) \in (\phi_0(x) + V_1) \]

for all \( x \in X \). For each \( x \in X \), let

\[ \phi_1(x) = (f_1(x) + V_1) \cap \phi_0(x) . \]
It is clear that $\phi_1(x)$ is non-empty and convex for all $x \in X$. Furthermore by Lemma 1 applied to $U_{V_1}$, $\phi_0$ and $f_1$, we have that $\phi_1$ is lower-semicontinuous.

Suppose that we have construct

$$\phi_2, \ldots, \phi_k \text{ and } f_1, \ldots, f_k.$$ 

If we apply Lemma 3 to

$$\phi = \phi_k \text{ and } V = V_{k+1},$$

there is a continuous map $f_{k+1}$ of $X$ into $Y$ such that

$$f_{k+1}(x) \in (\phi_k(x) + V_{k+1})$$

for all $x \in X$.

If $x \in X$, let

$$\phi_{k+1}(x) = (f_{k+1}(x) + V_{k+1}) \cap \phi_k(x).$$

It is clear that $\phi_{k+1}(x)$ is non-empty and convex for all $x \in X$. By Lemma 1 applied to

$$U_{V_{k+1}}, \phi_k \text{ and } f_{k+1},$$

we have that $\phi_{k+1}$ is lower-semicontinuous. This step completes the construction by induction of the sequences $\{\phi_n\}_{n \in \mathbb{N}}$ and $\{f_n\}_{n \in \mathbb{N}}$.

We show that for every

$$x \in X, \{\phi_n(x) | n \in \mathbb{N}\}$$

is a base for a Cauchy filter in

$$(\phi(x), \mathcal{U}_{\phi(x)}).$$

In fact, from

$$\phi_{k+1}(x) \subseteq \phi_n(x)$$
and \( \phi_n(x) \) is non-empty for all \( n \in \mathbb{N} \), we have \( \{ \phi_n(x) \mid n \in \mathbb{N} \} \) is a base for a filter \( F_x \) in \( \phi(x) \).

Let \( U \in \mathcal{U} \). The set \( \{ \phi(y) \}_{y \in X} \) is equimetrizable by means of \( \mathcal{U} \), and hence there exists \( n_0 \in \mathbb{N} \) such that

\[
(\phi(y) \times \phi(y)) \cap U_{n_0} \subseteq U
\]

for all \( y \in X \). Then

\[
\phi_{n_0 + 1}(x) \times \phi_{n_0 + 1}(x) \subseteq U \cap (\phi(x) \times \phi(x)):
\]

if

\[
(y, z) \in \phi_{n_0 + 1}(x) \times \phi_{n_0 + 1}(x) \subseteq \phi(x) \times \phi(x),
\]

\[
y = f_{n_0 + 1}(x) + v_{n_0 + 1} \quad \text{and} \quad z = f_{n_0 + 1}(x) + v'_{n_0 + 1},
\]

where

\[
v_{n_0 + 1}, v'_{n_0 + 1} \in V_{n_0 + 1}.
\]

So

\[
y - z = v_{n_0 + 1} - v'_{n_0 + 1} \in V_{n_0}
\]

and hence

\[
(y, z) \in U_{V_{n_0}}.
\]

Therefore

\[
(y, z) \in U \cap (\phi(x) \times \phi(x)).
\]

But for each \( x \in X \), \( F_x \) is a Cauchy filter in the complete uniform space

\[
(\phi(x), \mathcal{U}_{\phi(x)}).
\]
and so there exists

\[ f(x) \in \lim_{\mathcal{U}} |_{\phi(x)} F_x \]

with

\[ f(x) \in \phi(x). \]

Now we can show that \( f \) is a continuous map of \( X \) into \( Y \). Let \( x_0 \in X \) and \( U \in \mathcal{U} \). There is an \( W \in \mathcal{U} \) open symmetric such that \( W \circ W \subset U \). For each \( x \in X \) and \( n \in \mathbb{N} \), let

\[ l_n(x) \in \phi_n(x) \cap (V_n + f(x)) \cdot (f(x) \in \phi_n(x)) \]

Then

\[ \lim_{n \to \infty} l_n(x) |_{n \in \mathbb{N}} = f(x). \]

As \( \{\phi(x)\}_{x \in X} \) is equimetrizable by means of \( \mathcal{U}_1 \), there exists \( n_0 \in \mathbb{N} \) such that

\[ (\phi(x) \times \phi(x)) \cap (U_{v_{n_0}} \circ U_{v_{n_0}} \circ U_{v_{n_0}}) \subset W \]

for all \( x \in X \). \( \phi_{n_0} \) is lower-semicontinuous and hence

\[ V_{n_0} = \{ x \in X | \phi_{n_0}(x) \cap W[f(x_0)] \neq \emptyset \} \]

is an open neighborhood of

\[ x_0 \ (f(x_0) \in \phi_{n_0}(x_0)). \]

But

\[ f(V_{n_0}) \subset U[f(x_0)]. \]

In fact for every \( x \in V_{n_0} \), let

\[ a_x \in \phi_{n_0}(x) \cap W[f(x_0)]. \]
Then
\[ a_x = f_{\nu_0}(x) + \nu_{\nu_0}, \]
where
\[ \nu_{\nu_0} \in V_{\nu_0}. \]
So
\[ f(x) - a_x = (f(x) - I_{\nu_0}(x)) + (I_{\nu_0}(x) - f_{\nu_0}(x)) - \nu_{\nu_0} \in V_{\nu_0} + V_{\nu_0} + V_{\nu_0}. \]
Hence
\[ (f(x), a_x) \in U_{\nu_0} + V_{\nu_0} + V_{\nu_0} = U_{\nu_0} \circ U_{\nu_0} \circ U_{\nu_0}, \]
and as
\[ (f(x), a_x) \in \phi(x) \times \phi(x), \]
we have that
\[ (f(x), a_x) \in W. \]
On the other hand
\[ (a_x, f(x_0)) \in W. \]
Hence
\[ (f(x_0), f(x)) \in W \circ W \subset U. \]
So
\[ f(x) \in U[f(x_0)]. \]
This proves that \( f \) is continuous and hence a selection of \( \phi \).
f) \( \Longrightarrow \) e).
It is sufficient to observe that while $Y$ is metrizable, every family of subsets of $Y$ is equimetrizable by means of the uniformity $\mathcal{U}$ of $Y$.

e) $\implies$ d).

It is enough to bear in mind that every Frechet space is metrizable, locally convex and complete, and every closed subset of a complete space is complete.

d) $\implies$ c).

This is a consequence of that every Banach space is a Frechet space.

c) $\implies$ b).

It suffices to know that every $l_1(A)$ is a Banach space and if $\text{card}(A) \leq c$, the separability degree of $l_1(A) \leq c$

$$(D = \{ y : A \rightarrow R \mid a \in A \mid y(a) \neq 0 \} \text{ is finite and } \text{im}(y) \subseteq Q \}$$

is dense in $l_1(A))$.

b) $\implies$ a).

Let $\mathcal{U} = \{ U_j \}_{j \in J}$ be an open cover of $X$ with $\text{card}(J) \leq c$. Consider the real Banach space $E = l_1(J)$. Let

$$C = \{ y \in E \mid y(j) \geq 0 \text{ for all } j \in J \text{ and } \sum_{j \in J} y(j) = 1 \}.$$

Then:

1) $C$ is convex.

2) $C$ is closed. In fact, for every $j \in J$, the map $p_j : l_1(J) \rightarrow R$ defined by $p_j(y) = y(j)$ for all $y \in l_1(J)$, is continuous. So

$$C = \left( \bigcap_{j \in J} p_j^{-1}([0, \infty)) \right) \cap S_1,$$

where

$$S_1 = \{ x \in E \mid \| x \| = 1 \},$$

is closed in $l_1(J)$. For each $x \in X$, we consider

$$\phi(x) = C \cap \{ y \in E \mid y(j) = 0 \text{ for all } j \in J \text{ such that } x \in U_j \}.$$
It happens that:

1) For all \( x \in X \), \( \phi(x) \) is non-empty and convex.
2) For all \( x \in X \), \( \phi(x) \) is closed.

III) For all \( y \in C \) and \( \xi > 0 \), there exists an \( y' \in C \) such that:

1) \( \| y - y' \| < \xi \).

2) There are \( j_1, \ldots, j_n \in J \) such that \( y'(j) = 0 \) for all

\[
\forall j \in J \setminus \{j_1, \ldots, j_n\}, \quad y'(j) > 0
\]

for all \( i = 1, \ldots, n \), and \( y(j_i) > 0 \) for all \( i = 1, \ldots, n \).

In fact, from \( y \in C \) we have

\[
\sum_{j \in J} y(j) = 1.
\]

Hence there are \( j_1, \ldots, j_n \in J \) such that

\[
y(j_1) > 0, \ldots, y(j_n) > 0
\]

and

\[
y(j_1) + \ldots + y(j_n) = \delta > 1 - \frac{\xi}{2}.
\]

Define \( y' \in C \) by

\[
y'(j) = \begin{cases} 
0 & \text{if } j \in J \setminus \{j_1, \ldots, j_n\}; \\
y(j_1) + (1 - \delta) & \text{if } j = j_k; \\
y(j_k) & \text{if } j = j_k, \quad k = 2, \ldots, n.
\end{cases}
\]

Then

\[
\| y - y' \| = \sum_{j \in J} |y(j) - y'(j)| = \sum_{j \notin \{j_1, \ldots, j_n\}} y(j) + 1 - \delta = 2(1 - \delta) < \xi
\]

Furthermore, by construction, it is clear that 2) holds.

IV) \( \phi \) is lower-semicontinuous.
By (A) it is enough to prove that for every
\[ x \in X, \ y \in \phi(x) \]
and \( \xi > 0 \), there is a neighborhood \( V^x \) of \( x \) in \( X \) such that for all \( x' \in V^x \) it holds that there exists
\[ y' \in \phi(x') \quad \text{with} \quad ||y - y'|| < \xi. \]

By III) there exists \( y' \in C \) such that \( ||y - y'|| < \xi \) and there are \( j_1, \ldots, j_n \in J \) such that
\[ y'(j_1) > 0, \ldots, y'(j_n) > 0, \quad y(j_1) > 0, \ldots, y(j_n) > 0 \]
and \( y'(j) = 0 \) for all \( i \in J - \{j_1, \ldots, j_n\} \).

Let
\[ V^x = U_{j_1} \cap \ldots \cap U_{j_n} \]
(if \( i = 1, \ldots, n \) then \( x \in U_{j_i} \) since \( y(j_i) > 0 \).) For every \( x' \in V^x \), \( y' \in \phi(x') \) since
\[ y'(j) = 0 \quad \text{if} \quad j \in J - \{j_1, \ldots, j_n\}. \]

So, \( \phi \) is lower-semicontinuous.

By the hypothesis there is a selection \( f \) of \( \phi \). For each \( j \in J \), define
\[ f_j : X \rightarrow \mathbb{R} \quad \text{by} \quad f_j = p_j \circ f. \]

Then:
A) \( f_j(x) \geq 0 \) for all \( j \in J, \ x \in X \) since \( f(x) \in C \).
B) For all \( j \in J \), \( f_j \) is continuous.
C) For all \( x \in X \),
\[ \sum_{j \in J} f_j(x) = \sum_{j \in J} f(x)(j) = 1, \]
since \( f(x) \in C \).
D) \( \sup_{\mathcal{A}} (f_j) \subseteq U_j \) for all \( j \in J \), because \( x \notin U_j \) implies \( f_j(x) = f(x)(j) = 0 \) \( (f(x) \notin \phi(x)) \).

By VII.8.6 4. of [11], there is a \( \{g_j\}_{j \in J} \) locally finite continuous partition of unity subordinated to

\[ \{ \sup_{\mathcal{A}} (f_j) \}_{j \in J} \]

and hence subordinated to \( \mathcal{U} \). So, by Proposition 2, \( X \) is normal and \( c \)-paracompact.

**Corollary 6** ([14], [1], [6]).—Let \( X \) be a topological space. The following are equivalent:

a) \( X \) is paracompact and normal.

b) For every set \( A \) and every

\[ \phi: X \longrightarrow 2^{h(A)} \]

lower-semicontinuous such that \( \phi(x) \) is closed and convex for all \( x \in X \), \( \phi \) admits a selection.

c) If \( Y \) is a real complex Banach space, every \( \phi: X \longrightarrow 2^Y \) lower-semicontinuous such that \( \phi(x) \) is closed and convex for all \( x \in X \), admits a selection.

d) If \( Y \) is a real or complex Fréchet space, every \( \phi: X \longrightarrow 2^Y \) lower-semicontinuous such that \( \phi(x) \) is closed and convex for all \( x \in X \), admits a selection.

e) If \( Y \) is a real or complex metrizable locally convex topological vector space, every \( \phi: X \longrightarrow 2^Y \) lower-semicontinuous such that \( \phi(x) \) is convex and complete for all \( x \in X \), admits a selection.

f) If \( Y \) is a real or complex locally convex topological vector space, every \( \phi: X \longrightarrow 2^Y \) lower-semicontinuous such that \( \phi(x) \) is convex and complete for all \( x \in X \) and \( \{\phi(x)\}_{x \in X} \) is equimetrizable by means of a uniformity in \( Y \), admits a selection.

**Proposition 7**.—Let \( X \) be a \( c \)-paracompact and normal topological space, \( A \) a closed subset of \( X \), \( F \) a real or complex Fréchet topological vector space with separability degree \( \leq c \), and \( f \) a continuous map of
A into $F$. Then there exists a continuous map $\tilde{f}$ of $X$ into $F$ such that

$$\tilde{f}|A = f \quad \text{and} \quad \tilde{f}(X) \subseteq \text{Conv}(f(A)),$$

(where $\text{Conv}(f(A))$ is the convex hull of $f(A)$).

**Proof.**—Consider $\phi : X \rightarrow 2^F$ defined by

$$\phi(x) = \begin{cases} \{f(x)\} & \text{if } x \in A \\ \text{Conv}(f(A)) & \text{if } x \notin A. \end{cases}$$

Then $\phi$ is lower-semicontinuous. Moreover $\phi(x)$ is closed and convex for all $x \in X$. So, by Lemma 5, there exists $\tilde{f}$, selection of $\phi$. It is clear that $\tilde{f}$ fulfils the requirements of the proposition.

*(Continuará.)*