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On the Boundedness of the Local Resolvent Function

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1. INTRODUCTION

Let $L(X)$ denote the Banach algebra of all (continuous linear) operators defined on a complex Banach space X.

Given an operator $T \in L(X)$, a complex number λ belongs to the *resolvent set* $\rho(T)$ of T if there exists $(\lambda - T)^{-1} =: R(\lambda, T) \in L(X)$. We denote by $\sigma(T) := \mathbb{C}\backslash\rho(T)$ the *spectrum* of T. The *resolvent map* $R(.,T) : \rho(T) \to L(X)$ is analytic. Moreover $R(\lambda, T)$ satisfies the following equation [3, Corollary 1.5]

$$\|R(\lambda, T)\| \geq \frac{1}{\operatorname{dist}(\lambda, \sigma(T))}\,. \tag{1}$$

Hence the resolvent operator is unbounded.

In this paper, we prove that if T is a *hyponormal* operator (i.e. $T^*T - TT^* \geq 0$) on a complex Hilbert space H and x is a nonzero vector $x \in H$ such that $\sigma(x, T)$ has empty interior, then the local resolvent function $\widehat{x}_T$ is unbounded. In particular, if $\sigma(T)$ has empty interior, then $\widehat{x}_T$ is unbounded for every nonzero x.

Our main result is Theorem 2, where we prove that for a normal operator T on a complex Hilbert space H, the local resolvent function $\widehat{x}_T$ is unbounded for every nonzero $x \in H$ if and only if the spectrum of T has empty interior. In particular, if T is selfadjoint, then $\widehat{x}_T$ is unbounded for every nonzero vector.

In Example 2, we show that one of the implications of this result does not hold in general for subnormal operators.

Moreover, in Example 1, we give an operator T in c_0 and a nonzero vector x so that $\widehat{x}_T$ is bounded although $\sigma(x, T)$ has empty interior.

In [4, Proposition 2.1] it was proved that for every operator T with the SVEP (see definition below) there is a vector $x \in X$ such that the local resolvent function $\widehat{x}_T$ is unbounded. However, [4, Proposition 2.2] contains an example of a (non-selfadjoint) normal operator N on a separable Hilbert space H and a vector $x \in H$ such that the local resolvent function $\widehat{x}_N$ is bounded. This example depends on the fact that the interior of $\sigma(N)$ is nonempty. Therefore, in [4] it is raised the question whether for a selfadjoint operator T, the local resolvent function is unbounded.

Finally, we observe that if the analytic capacity (see definition in Section 2) of the local spectrum is zero, the local resolvent function is unbounded.

Now we recall some concepts and notations of local spectral theory. We refer to [2] for further information.

Given $T \in L(X)$, a complex number λ belongs to the *local resolvent set* of T at x, denoted $\rho(x,T)$, if there exists an analytic function $w : U \to X$, defined on a neigborhood U of λ, which satisfies $(\mu - T)w(\mu) = x$, for every $\mu \in U$. The *local spectrum set* of T at x is $\sigma(x,T) := \mathbb{C} \setminus \rho(x,T)$.

An operator $T \in L(X)$ satisfies the *Single Valued Extension Property* (hereafter referred to as SVEP) if $(\lambda - T)h(\lambda) = 0$ implies $h \equiv 0$ for any analytic solution defined on any open subset of the plane with values in X.

Related with the boundedness of the local resolvent function, Clancey [1, Theorem I.3.1] proved that every hyponormal operator without eigenvalues on a complex separable Hilbert space H and unit vector $x \in H$ satisfy the following inequality

$$\|\widehat{x}_T(\lambda)\| \leq \frac{1}{\operatorname{dist}(\lambda, \sigma(x,T))}\,.$$

Our first main result (Theorem 2) proves that this inequality could be strict even for normal operators.

2. Main results

First we give a sufficient condition for the unboundedness of the local resolvent functions of a hyponormal operator.

THEOREM 1. *Let T be hyponormal operator on a complex Hilbert space H and let $x \in H \setminus \{0\}$. If $\sigma(x,T)$ has empty interior, then the local resolvent function $\widehat{x}_T$ is unbounded.*

The following Corollary answers the question in [4, Remark 2.3] about the boundedness of the local resolvent functions for a selfadjoint operator.

COROLLARY 1. *Let T be a selfadjoint operator on a complex Hilbert space H. Then the local resolvent function is unbounded for every nonzero $x \in H$.*

COROLLARY 2. *If T is a hyponormal operator on a complex Hilbert space H and $\sigma(T)$ has empty interior, then the local resolvent function $\widehat{x}_T$ is unbounded for every nonzero $x \in H$.*

The proof of Theorem 1 gives the following result.

PROPOSITION 1. *Let X is a complex reflexive Banach space and let $T \in L(X)$ be an operator having the SVEP. Then for every vector $x \in X \setminus \bigcap_{\lambda \in \mathbb{C}} R(\lambda - T)$ such that $\sigma(x, T)$ has empty interior, the local resolvent function $\widehat{x}_T$ is unbounded.*

The following example show that some condition on the space X is necessary in Proposition 1.

EXAMPLE 1. Let V be the right shift on the space $c_0(\mathbb{Z})$ of all "doubly infinite" null sequences $(a_i)_{i=-\infty}^{\infty}$, endowed with the supremum norm $\|\cdot\|_\infty$.
The vector e_0 in the unit vector basis of $c_0(\mathbb{Z})$ satisfies that $\sigma(e_0, V) = \partial\mathbb{D}$ and the local resolvent function $\widehat{e_0}_V$ is bounded in $\rho(e_0, V)$, although $\sigma(e_0, V)$ has empty interior.

Next, we characterize when the normal operators admitting bounded local resolvents.

THEOREM 2. *Let N be a normal operator on a complex Hilbert space H. Then the spectrum of N has empty interior if and only if the local resolvent function is unbounded for every nonzero $x \in H$.*

COROLLARY 3. *Let N be a normal operator on a complex Hilbert space H. If the closure of $\operatorname{int}(\sigma(N))$ coincides with $\sigma(N)$, then there exists a vector $x \in H$ such that $\sigma(x, N) = \sigma(N)$ and $\widehat{x}_N$ is bounded.*

Now we give an example showing that Theorem 2 is not valid for subnormal operators.

EXAMPLE 2. Let U be the unilateral (isometric) shift on $\ell^2(\mathbb{N})$. This operator is a restriction of the (unitary) bilateral shift on $\ell^2(\mathbb{Z})$. Therefore it is subnormal. Moreover, the local resolvent function $\widehat{x}_U$ is unbounded for every nonzero x.

Final Remarks. (a) In Example 2 we could have derived the fact that he local resolvent function $\widehat{x}_U$ is unbounded from a general argument. A simple transformation allows us to see $\widehat{x}_U$ as an analytic function on the unit disc $\mathbb{D}$ with values on the Hilbert space ℓ_2. Now, for any bounded analytic function

$$f : z \in \mathbb{D} \longrightarrow \sum_{n=0}^{\infty} a_n z^n \in \ell_2$$

we have that $\lim_{n\to\infty} \|a_n\| = 0$ because ℓ_2 has the analytic Radon-Nikodym property.

(b) Let K be a compact subset of the complex plane $\mathbb{C}$ and let us denote by Ω the complement of K in the Riemann sphere $\mathbb{C}_\infty$. Recall that the *analytic capacity* of K is defined by

$$\gamma(K) := \sup\{|f'(\infty)| : f \in H^\infty(\Omega),\ |f|_\Omega \leq 1\},$$

where $H^\infty(\Omega)$ stands for the algebra of all bounded analytic functions on Ω.

If $T \in L(X)$ has the SVEP and $x \in X \setminus \{0\}$ satisfies $\gamma(\sigma(x, T)) = 0$, then $\widehat{x}_T$ is unbounded. In particular, if $\sigma(T)$ is countable, like in the case of a compact operator, then $\widehat{x}_T$ is unbounded for every nonzero x.

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