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GENERALIZED LINS-MANDEL SPACES
AND BRANCHED COVERINGS OF $S^2$
It is well known that PL-manifolds are representable by edge-coloured graphs. This combinatorial approach turns out to be a useful tool for generating and investigating wide classes of manifolds represented by “highly-symmetric” graphs (see, for example, [6], [11], [14] and [46]). In particular, Lins-Mandel spaces ([29]) have been intensively studied in the last ten years. In this paper, after a short survey on the related research areas, we describe recent results (mainly obtained in the Ph.D. thesis [40] of one of the authors) about the topological structure of Lins-Mandel spaces in terms of branched coverings of $S^3$. We also illustrate generalizations of these spaces and their relations with other representation theories.

1. Branched coverings of $S^3$

With the term manifold we always mean a compact, connected, orientable PL-manifold without boundary.

Let $\tilde{M}, M$ be triangulated n-manifolds and let $N$ be an $(n-2)$-subcomplex of $M$; a non-degenerate map $f: \tilde{M} \rightarrow M$ is a $d$-fold covering map, branched over $N$, if:

- $f' = f |_{\tilde{M} - f^{-1}(N)}$ is an ordinary covering of $M - N$ of degree $d$;
- $N = \{ x \in M | \#f^{-1}(x) < d \}$ (branching set).

$\tilde{M}$ is said to be a branched covering of $M$.

$$\begin{CD}
\tilde{M} - f^{-1}(N) @>i>> \tilde{M} \\
\downarrow f' @. \downarrow f \\
M - N @>i>> M
\end{CD}$$

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A remarkable result by R.H. Fox ([18]) states that a branched covering is uniquely determined by the ordinary covering induced by restriction. This proves the existence of a one-to-one correspondence between the $d$-fold coverings of $M$ branched over $N$ and the equivalence classes of monodromies (i.e. transitive representations) $\omega : \pi_1(M - N) \to \Sigma_d$, where $\Sigma_d$ denotes the symmetric group on $d$ elements. Moreover, the Fox theorem gives the possibility of extending the concepts of regular or cyclic coverings to branched coverings. General references on the subject are [2], [18], [35] and [42].

The notion of branched covering can be extended to more general spaces (see [18] and [32]), including quasi-manifolds\(^1\).

Branched coverings of spheres are of great interest, in particular as a method for representing manifolds. A classical result in this direction goes back to J.W. Alexander:

**Proposition.** [1] Every $n$-manifold is a covering of $S^n$ branched over the $(n-2)$-skeleton of a standard $n$-simplex. \(\square\)

Refinements of the Alexander theorem can be investigated in two different directions:

i) minimalize the number of sheets;

ii) find universal branching sets.

In dimension three, these approaches respectively lead to the possibility of representing all 3-manifolds by means of coloured knots and transitive permutation pairs.

In the first direction define a **coloured knot** as a pair $(L, \omega)$, where $L$ is a knot and $\omega : \pi_1(S^3 - L) \to \Sigma_3$ is a simple monodromy. In fact, $(L, \omega)$ can be drawn by colouring the arcs of a suitable diagram of $L$ by $0, 1, 2$, so that the arc $\alpha$ is coloured $c$ iff $\omega(\alpha)$ fixes $c$.

**Representation theorem 1.** [24], [33] Every 3-manifold is a simple 3-fold covering of $S^3$, branched over a knot. Thus, every 3-manifold is representable by coloured knots. \(\square\)

In the second direction, consider the handcuff-graph $G$ embedded in $S^3$ as in Figure 1.

\[\text{FIGURE 1}\]

Since $\pi_1(S^3 - G)$ is a free group with two generators, the monodromy of any covering

\(^1\)A quasi-manifold is a pseudo-manifold in which the star of every simplex is strongly connected ([19]).
GENERALIZED LINS-MANDEL SPACES AND BRANCHED COVERINGS OF $S^3$

$\tilde{M}$ of $S^3$, branched over $G$, is given by a transitive permutation pair $(\sigma, \tau)$ of $\Sigma_6$, $b$ being the degree of the covering. We will denote $\tilde{M}$ by $N_b(\sigma, \tau)$.

**Representation theorem 2.** [34] *Every 3-manifold is a covering of $S^3$, branched over the graph $G$. Therefore, every 3-manifold is representable by transitive permutation pairs.* $\square$

**REMARK.** It seems to be interesting to relate these two representation theories. In particular, the possibility of obtaining a coloured knot representing a 3-manifold $M$, starting from a pair $(\sigma, \tau)$ representing $M$, would give a combinatorial proof of Hilden-Montesinos theorem. In the opposite direction, the problem has been solved ([22]) by producing an algorithm for finding a transitive permutation pair $(\sigma, \tau)$ representing $M$, starting from a coloured knot $(L, \omega)$ representing $M$. As a consequence, if $L$ admits a “coloured diagram” with $n$ crossings, then $M$ is a $3n$-fold covering of $S^3$, branched over $G$.

2. EDGE-COLOURED GRAPHS AND GEMS

An $(n+1)$-coloured graph is a pair $(\Gamma, \gamma)$, where:

- $\Gamma$ is a finite connected regular multigraph of degree $n+1$;
- $\gamma : E(\Gamma) \rightarrow \Delta_n = \{0, 1, \ldots, n\}$ is a proper edge-coloration (i.e. adjacent edges have different colours).

Every $(n+1)$-coloured graph represents a pseudosimplicial complex ([26]) $K(\Gamma)$ defined by:

- taking an $n$-simplex $\sigma(x)$ for each vertex $x \in V(\Gamma)$ and labelling its vertices by the elements of $\Delta_n$;
- identifying, for every pair $x, y \in V(\Gamma)$ of $c$-adjacent vertices, the $(n-1)$-faces of $\sigma(x)$ and $\sigma(y)$ opposite to the vertices labelled by $c$.

The underlying space $|K(\Gamma)|$ is a quasi-manifold which is orientable if $\Gamma$ is bipartite ([17]). In dimension three, quasi-manifolds are also called singular manifolds ([34]).

If $B \subset \Delta_n$, $\#B = h \leq n$, there is a bijection between the components of the partial graph $\Gamma_B = (V(\Gamma), \gamma^{-1}(B))$ (called $h$-residues) and the $(n-h)$-simplices of $K(\Gamma)$ whose vertices are labelled by $\Delta_n - B$.

An $n$-gem is an $(n+1)$-coloured graph representing an $n$-manifold; every manifold $M$ is representable by gems ([17], [41]).
Edge-coloured graph techniques provide a combinatorial way for representing manifolds; for general references see [3], [17], [29] and [47].

**Remark.** An algorithm for obtaining, from a bipartite crystallization\(^2\) \(\Omega\), a transitive permutation pair \((\sigma, \tau)\) such that \(|K(\Omega)| \cong N(\sigma, \tau)\), is contained in [12] and [44]. The algorithm has been extended to the general case of bipartite 4-coloured graphs in [23].

A colour-preserving morphism (cp-morphism) between \((n + 1)\)-coloured graphs

\[ f : (\Gamma, \gamma) \rightarrow (\Gamma', \gamma') \]

naturally induces a map

\[ K(f) : |K(\Gamma)| \rightarrow |K(\Gamma')| \]

between the associated underlying spaces.

A cp-morphism \(f : (\Gamma, \gamma) \rightarrow (\Gamma', \gamma')\) is said to be an \(m\)-covering \((1 \leq m \leq n)\) if the restriction of \(f\) to the \(m\)-residues is one-to-one ([45]). In particular:

- if \(m = n\), then \(K(f)\) is an ordinary covering,
- if \(m = 1\), then \(K(f)\) is a branched covering; moreover, if \(|K(\Gamma')|\) is a manifold, the branching set is given by the \((n - 2)\)-simplices of \(K(\Gamma')\) represented by the 2-residues (bicoloured cycles) of \((\Gamma', \gamma')\) not ordinarily covered by \(f\) (i.e., the 2-residues whose counterimages via \(f\) have at least one component non-isomorphic to it).

### 3. LINS-MANDEL GRAPHS AND SPACES

The family of Lins-Mandel 4-coloured graphs

\[ \Phi = \{G(b, l, t, c) \mid b, l \in \mathbb{Z}^+, t \in \mathbb{Z}_{2t}, c \in \mathbb{Z}_b\} \]

is defined in [29] by the following rules: the set of vertices of \(G(b, l, t, c)\) is

\[ V = \mathbb{Z}_b \times \mathbb{Z}_{2l} \]

and the coloured edges are obtained by these four fixed-point-free involutions on \(V:\)

\(^2\)A crystallization is a gem with exactly \((n + 1)\) \(n\)-residues ([17]).
\[ t_0(i, j) = (i, -j + (-1)^j), \]
\[ t_1(i, j) = (i, j + (-1)^j), \]
\[ t_2(i, j) = (i + \eta(j), 1 - j), \]
\[ t_3(i, j) = (i + \eta(j - t), 1 - j + 2t); \]

where \( \eta : \mathbb{Z}_{2l} \to \{-1, 1\} \) is defined by
\[
\eta(j) = \begin{cases} 
+1 & \text{if } 1 \leq j \leq l \\
-1 & \text{otherwise}
\end{cases}
\]

For each \( k \in \Delta_3 \), we join the vertices \( v, w \in V \) by a \( k \)-coloured edge iff \( t_k(v) = w \).

Roughly speaking, the graph \( G(b, 1, t, e) \) is obtained by taking \( b \) copies \( C_i \) \((i \in \mathbb{Z}_b)\) of the \([0, 1]\)-cycle of length \( 2l \) (so that \( V(C_i) = \{(i, j) \mid j \in \mathbb{Z}_{2l}\} \)) joined with:

- \( C_{i-1} \) and \( C_{i+1} \) by the 2-coloured edges,
- \( C_{i-c} \) and \( C_{i+c} \) by the 3-coloured edges.

**Figure 2**

Each \( G(b, l, t, c) \in \mathcal{S} \) is connected and bipartite; hence, it represents a (connected, orientable) singular 3-manifold \( S(b, l, t, c) \). These spaces have been introduced as a combinatorial generalization of the lens spaces; in fact, \( G(2, l, t, 1) \) is the “normal” graph representing the lens space \( L(l, t) \) ([15]). They have been intensively studied by M.R.Casali, A.Cavicchioli, A.Donati, L.Grasselli, D.L.Johnson-R.M.Thomas and, of course, S.Lins-A.Mandel (see [5], [6], [7], [8], [9], [10], [13], [21], [28] and [29]). We summarize the main results of these works:

1) if \((l, t) = 1\) and one of the following conditions holds:
   - \( l \) is even and \( c = \pm 1 \),
   - \( l \) is odd and \( c = (-1)^l \),
then \( S(b, l, t, c) \) is a 2-fold covering of \( \mathbb{S}^3 \), branched over a link;

2) \( S(b, l, l - 1, 1) \cong S(b, l, 1, -1) \) is the 2-fold covering of \( \mathbb{S}^3 \), branched over the torus link of type \([b, l]\), i.e. the Brieskorn manifold \( M(b, l, 2) \); if \( b \) and \( l \) are odd and coprime, these spaces are Seifert fibered homology spheres of Heegaard genus 2.
By making use of the symmetry of the Lins-Mandel graphs, it is easy to prove the existence of homeomorphisms between the associated spaces; as a consequence, we can restrict the range of some parameters, without loss of generality, as stated below.

(A) $S(b, l, t, c) \cong S(b, kl, kt, c) \Rightarrow (l, t) = 1$

(B) $S(b, l, t, c) \cong S(b, l, t - l, -c) \Rightarrow 1 \leq t \leq l$

(C) $S(b, l, t, c) \cong S(b, l - t, -c) \Rightarrow l \text{ odd}$

Thus, from now on, we restrict our attention to the subfamily

$$\tilde{\mathcal{G}} = \{G(b, l, t, c) \in \mathcal{G} | (l, t) = 1, 1 \leq t \leq l, t \text{ odd}\}.$$ 

The cases (i) $c = 0$ and (ii) $l = 1, c = \pm 1$ are "trivial", because the graphs admit planar regular embeddings ([20]), and therefore $S(b, l, t, 0) \cong S^3 \cong S(b, 1, 1, -1)$ (see [16]).

Since a singular 3-manifold is a 3-manifold iff its Euler characteristic vanishes ([43]), a (rather complicated) calculation gives a complete characterization of the Lins-Mandel graphs representing manifolds:

**Proposition.** [37] A Lins-Mandel graph $G(b, l, t, c) \in \tilde{\mathcal{G}}$ represents a 3-manifold iff either $l$ is even or $c = 0, -1$. $\square$

The main result concerning the topological properties of the Lins-Mandel spaces is the following:

**Theorem.** [38] Let $G(b, l, t, c) \in \tilde{\mathcal{G}}$ and $S(b; l, t, c)$ be the associated Lins-Mandel space.

(a) $S(b, l, t, c)$ is a $b$-fold branched cyclic covering of $S^3$.

(b) Suppose $c \neq 0$ (recall that the case $c = 0$ is trivial). If $S(b, l, t, c)$ is a manifold, then the branching set is the 2-bridge link $b(l, t)$; otherwise, the branching set is a $\theta$-graph (Figure 3.a), non-trivially embedded when $l \neq 1$.

**Sketch of proof.** (a) The map $f_b: G(b, l, t, c) \to G(1, l, t, 0)$, defined by $f_b(i, j) = (0, j)$, is the 1-covering induced by the action of the cyclic group $\mathbb{Z}_b$, generated by the epimorphism $(i, j) \mapsto (i + 1, j)$, on $G(b, l, t, c)$. Therefore, the associated map $K(f_b): S(b, l, t, c) \to S(1, l, t, 0) \cong S^3$ is a branched cyclic covering map.

(b) When $S(b, l, t, c)$ is a manifold, it is easy to check that the set of the 2-residues of $G(1, l, t, 0)$ not ordinarily covered by $f_b$ does not depend on $b$ and $c$; moreover it contains
exactly four cycles. Incidence arguments on the lattice of the residues of $G(1, l, t, 0)$ show that the 1-subcomplex $R$, whose edges are represented by these four cycles, is homeomorphic to $S^1$ (resp. to $S^1 \coprod S^1$) iff $l$ is odd (resp. is even). Since $G(1, l, t, 0)$ and $R$ only depend on $l$ and $t$, $R$ is the branching set of the 2-fold covering $K(f_2) : S(2, l, t, 1) \cong L(l, t) \to S^3$.

The unicity of the representation of the lens spaces as 2-fold coverings of $S^3$ ([27]) proves that the branching set is the two-bridge link $b(l, t)$ (see [4]). For “non-manifolds”, the branching set is represented by five 2-residues and is homeomorphic to a $\theta$-graph; again, the embedding only depends on $l$ and $t$ (the description of the embedding can be found in [39]).

As a consequence of this theorem and of the Smith conjecture ([36]), we can completely characterize the spheres among Lins-Mandel spaces:

**Corollary.** [38] A Lins-Mandel manifold $S(b, l, t, c)$ is homeomorphic to $S^3$ iff either $c = 0$ or $l = 1$.

**Remark.** The monodromy $\omega : \pi_1(S^3 - b(l, t)) \to \Sigma_b$, associated to the covering, is defined by

- $\omega(m_1)(i) = i + 1$,
- $\omega(m_2)(i) = i - c$;

where $m_1$ and $m_2$ are meridians associated to the two bridges of $b(l, t)$ (see Figure 4).

Note that, if $l$ is odd, $b(l, t)$ is a knot and therefore $m_1$ and $m_2$ are associated to the same component of the branching set; this explains why $l$ odd implies $c = -1$. 

Let now $M(L, \omega)$ denote the $b$-fold cyclic covering of $S^3$, branched over an oriented link $L$, and defined by the monodromy $\omega : \pi_1(S^3 - L) \to \Sigma_b$. The following notions are given in [30]:

**Figure 3**

As a consequence of this theorem and of the Smith conjecture ([36]), we can completely characterize the spheres among Lins-Mandel spaces:

**Corollary.** [38] A Lins-Mandel manifold $S(b, l, t, c)$ is homeomorphic to $S^3$ iff either $c = 0$ or $l = 1$. □

**Remark.** The monodromy $\omega : \pi_1(S^3 - b(l, t)) \to \Sigma_b$, associated to the covering, is defined by

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Note that, if $l$ is odd, $b(l, t)$ is a knot and therefore $m_1$ and $m_2$ are associated to the same component of the branching set; this explains why $l$ odd implies $c = -1$. 

**Figure 4**
M(L, ω) is strictly-cyclic if ω(mi) = ω(mj), for every meridian pair mi, mj;

M(L, ω) is almost-stricly-cyclic if ω(mi) = ω(mj)±1, for every meridian pair mi, mj;

M(L, ω) is meridian-cyclic if ord(ω(mi)) = b, for every meridian mi;

M(L, ω) is singly-cyclic if there exists a meridian m such that ord(ω(mi)) = b.

It is straightforward that:

(a) ⇒ (b) ⇒ (c) ⇒ (d) ⇒ cyclic;

moreover, the five notions are equivalent when L is a knot.

As a direct consequence of the above theorem and remark, we obtain:

Corollary. [38] The Lins-Mandel manifold S(b, l, t, c) is a branched singly-cyclic covering of S^3. In particular:

- if c = -1, the covering is strictly-cyclic;
- if c = ±1, the covering is almost-strictly-cyclic;
- if (b, c) = 1, the covering is meridian-cyclic.

REMARKS. (1) The manifolds S(b, l, t, c) with (b, c) = 1 are precisely the Lins-Mandel manifolds whose corresponding gems G(b, l, t, c) are crystallizations ([5]).

(2) The Minkus manifolds M_n(k, h), investigated in [31], are particular cases of Lins-Mandel manifolds. Actually, we have M_n(k, h) ≃ S(n, k, h, −1).

To end this section, we present necessary and sufficient conditions for the isomorphism between Lins-Mandel gems. As a consequence, we get sufficient conditions for the homeomorphism between Lins-Mandel manifolds (different from the sphere and lens spaces).

Proposition. Let G = G(b, l, t, c), G' = G(b', l', t', c') ∈  be gems, with l, l' > 2. Then G is isomorphic to G' iff either

b' = b, l' = l, t' = ±t±1, c' = c^s

or

b' = b, l' = l, t' = ±t±1 + l, c' = −c^s;

where

s = \begin{cases} 
+1 & \text{if } (b, c) \neq 1 \\
±1 & \text{if } (b, c) = 1 
\end{cases}.
Hence, if one of the above conditions holds, the manifolds \( S(b, l, t, c) \) and \( S(b', l', t', c') \) are homeomorphic. \( \square \)

4. GENERALIZED LINS-MANDEL SPACES

The Lins-Mandel family only contains singly-cyclic coverings. This makes natural the attempt of extending it, in order to obtain the whole class of cyclic coverings of \( S^3 \), branched over two-bridge links.

We define a new class of 4-coloured graphs depending on five parameters:

\[ \mathfrak{G} = \{ \tilde{G}(b, l, t, c, c') \mid b, l \in \mathbb{Z}^+, \ t \in \mathbb{Z}_{2l}, \ c, c' \in \mathbb{Z}_b, \ \gcd(b, c, c') = 1 \}. \]

Each \( \tilde{G}(b, l, t, c, c') \) is defined by the following four fixed-point-free involutions on the set \( V = \mathbb{Z}_b \times \mathbb{Z}_{2l} \):

\[ \begin{align*}
\tilde{i}_0 &= \iota_0, \\
\tilde{i}_1 &= \iota_1, \\
\tilde{i}_2(i, j) &= (i + c' \eta(j), 1 - j), \\
\tilde{i}_3 &= \iota_3; 
\end{align*} \]

it represents a (connected, orientable) singular 3-manifold \( \tilde{S}(b, l, t, c, c') \).

We easily obtain the following results:
1) \( \tilde{S}(b, l, t, c, 1) \cong S(b, l, t, c) \);
2) \( \tilde{S}(b, l, t, 0, c') \cong \tilde{S}(b, l, t, c, 0) \cong \tilde{S}(b, 1, 1, -c', c') \cong S^3 \);
3) \( \tilde{S}(b, l, t, c, c') \cong \tilde{S}(b, l, t, c', c) \);
4) If \( (b, c) = 1 \) or \( (b, c') = 1 \), then there exists \( c'' \in \mathbb{Z}_b \) such that \( \tilde{S}(b, l, t, c, c') \cong S(b, l, t, c'') \); thus, the generalization is effective for the cases \( (b, c) \neq 1 \neq (b, c') \);
5) As in Section 3, we can assume, without loss of generality, \( (l, t) = 1, \ 1 \leq t \leq l \) and \( t \) odd.

The characterization of the manifolds among Lins-Mandel generalized spaces is similar to the previous one:
Proposition. [38] The graph $\tilde{G}(b, l, t, c, c')$ represents a 3-manifold iff either $l$ is even or at least one of the following conditions holds: (i) $c = 0$, (ii) $c' = 0$, (iii) $c = -c'$. □

Theorem. [38] The 3-manifold $\tilde{S}(b, l, t, c, c')$, with $c \neq 0 \neq c'$, is the $b$-fold cyclic covering of $S^3$, branched over $b(l, t)$. The associated monodromy is defined by:

$$\omega(m_1)(i) = i + c',
\omega(m_2)(i) = i - c.$$

Therefore, the class of generalized Lins-Mandel manifolds $\tilde{S}(b, l, t, c, c')$, with $c \neq 0 \neq c'$, is precisely the class of all cyclic coverings of $S^3$ branched over the two-bridge links (with the exception of the trivial link with two components). □

Corollary. [38] The 3-manifold $\tilde{S}(b, l, t, c, c')$ is a sphere iff either $c = 0$ or $c' = 0$ or $l = 1$. □

5. FURTHER GENERALIZATIONS

The attempt of obtaining a class of gems representing all coverings of $S^3$, branched over two-bridge links, leads to the following extension.

Take $b, l \in \mathbb{Z}^+$, $t \in \mathbb{Z}_{2l}$, with $(l, t) = 1$. Let $(\sigma, \tau)$ be a transitive permutation pair of $\Sigma_b$. Define the following four fixed-point-free involutions on $V = \mathbb{Z}_b \times \mathbb{Z}_{2l}$:

$$\bar{i}_0 = \iota_0,$$
$$\bar{i}_1 = \iota_1,$$
$$\bar{i}_2(i, j) = (\sigma^{\varphi(j)}(i), 1 - j),$$
$$\bar{i}_3(i, j) = (\tau^{-\varphi(j-t)}(i), 1 - j + 2t).$$

Denote by $\tilde{G}(b, l, t, \sigma, \tau)$ the resulting 4-coloured graph and by $\tilde{S}(b, l, t, \sigma, \tau)$ the associated space.
Lemma. \[39\] a) If \( v \) is the cyclic permutation \((0 1 2 \cdots b - 1)\), then \( \mathcal{S}(b, l, t, v', v^{-1}) \cong \mathcal{S}(b, l, t, c, c') \);

b) If \( \sigma = 1 \) or \( \tau = 1 \), then \( \mathcal{S}(b, l, t, \sigma, \tau) \cong \mathcal{S}^3 \);

c) \( \mathcal{S}(b, l, t, \sigma^{-1}, \tau^{-1}) \cong \mathcal{S}(b, l, t, \sigma, \tau) \cong \mathcal{S}(b, l, t, \sigma, \tau) \).

Let \( \varphi \) be the permutation \( \varphi = \sigma^{(t)} \tau^{(\varphi)} \).

Theorem. \[39\] \( \mathcal{S}(b, l, t, \sigma, \tau) \) is the \( b \)-fold covering of \( \mathcal{S}^3 \) branched over a 1-subcomplex \( R \) with the following description:

1) if \( l \) is odd and

a) \( \sigma, \tau, \varphi \neq 1 \), then \( R \) is a \( b \)-graph (Figure 3.a), non-trivially embedded when \( l \neq 1 \);

b) \( \sigma, \tau \neq 1 \) and \( \varphi = 1 \), then \( R \) is the two-bridge knot \( b(l, t) \);

c) \( \sigma = 1 \) or \( \tau = 1 \), then \( R \) is the trivial knot;

2) if \( l \) is even and

a) \( \sigma, \tau, \varphi \neq 1 \), then \( R \) is a handcuff-graph (Figure 3.b), non-trivially embedded;

b) \( \sigma, \tau \neq 1 \) and \( \varphi = 1 \), then \( R \) is the two-bridge link \( b(l, t) \);

c) \( \sigma = 1 \) or \( \tau = 1 \), then \( R \) is the trivial knot.

In any case, the fundamental group \( \pi_1(\mathcal{S}^3 - R) \) admits a presentation with two generators \( X, Y \) and, in the cases 1.a and 2.a, it is the free group \( \langle X, Y ; \emptyset \rangle \). The monodromy associated to the covering is defined by

\[
\omega(X) = \sigma,
\]

\[
\omega(Y) = \tau.
\]

REMARKS. (a) If \( l = 2 \) and \( t = 1 \), the branching set \( R \) is precisely the universal graph \( G \) of Montesinos (Figure 1). Moreover, \( \omega \) is the monodromy of the branched covering \( N_b(\sigma, \tau) \);

so we have the homeomorphism \( \bar{S}(b, 2, 1, \sigma, \tau) \cong N_b(\sigma, \tau) \). Since each (singular) 3-manifold is homeomorphic to a suitable \( N_b(\sigma, \tau) \), the subclass of graphs \( \{ \bar{G}(b, 2, 1, \sigma, \tau) \} \) is a (very symmetric) "universal" class of 4-coloured graphs representing all singular 3-manifolds.

(b) The class \( \mathcal{G}_{l,t} = \{ \bar{G}(b, l, t, \sigma, \tau) \mid \sigma, \tau \neq 1, \varphi = 1, b \in \mathbb{Z}^+ \} \) precisely represents all coverings of \( \mathcal{S}^3 \), branched over the two-bridge link \( b(l, t) \). Thus, every space of \( \mathcal{G}_{l,t} \) is a
manifold. Moreover, since a non-toroidal two-bridge link is universal ([25]), every subclass
\(\{\bar{\Theta}_t, | t \neq \pm 1 \text{ (mod } l)\}\) is a “universal” class of gems representing all 3-manifolds.

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Figure 1. The Montesinos universal graph $G$
Figure 2. The Lins-Mandel gem $G(4, 4, 1, 3)$
Figure 3. (a) A \(\theta\)-graph (trivially embedded) — (b) A handcuff-graph (trivially embedded)
Figure 4. The two-bridge knot $b(3,1)$
Figure 5. The graph $\tilde{G}(6,3,2,3,2)$
Figure 6. $\tilde{O}(6, 2, 1, (0 1 2)(3 4 5), (0 2 5 3)(1 4))$ represents $S^2 \times S^1$. 