Periodic solutions of a non coercive hamiltonian system

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Received June 22, 1994. Revised May 4, 1995

Abstract
In this paper, we study the existence of periodic solutions of hamiltonian systems:
\[ \dot{x} = JH'(t, x) \]
where the hamiltonian \( H \) is non coercive of the type
\[ H(t, r, p) = f(|p - Ar|) + h(t) \cdot (r, p). \]

1. Introduction
Let be given a relativistic particle with charge \( e \) and mass at rest \( m_0 \) submitted to
a constant uniform magnetic field \( B \) and a uniform electric field \( E(t) \), its movement
is governed by the hamiltonian equations:
\[ \dot{r} = \frac{\partial H}{\partial p}(t, r, p), \quad \dot{p} = -\frac{\partial H}{\partial r}(t, r, p) \]
where \( H \) is the particle energy given in terms of the time \( t \), the position \( r \) and the
impulsion \( p \) by the formula:
\[ H(t, r, p) = c \left[ m_0^2 c^2 + \left| p - \frac{e}{2c} B \land r \right|^2 \right]^{1/2} - e E(t) \cdot r \]
where \( c \) is the light speed.
This leads us to study the existence of periodic solutions of the hamiltonian systems:

\[(H) \quad \dot{x} = JH'(t, x)\]

where the hamiltonian \( H \) is non coercive of the type

\[H(t, r, p) = f(|p - Ar|) + h(t) \cdot (r, p)\]

with \( A \) a matrix of order \( n \), \( f : \mathbb{R}_+ \rightarrow \mathbb{R} \) is a non decreasing convex continuously differentiable function, \( h : \mathbb{R} \rightarrow \mathbb{R}^{2n} \) is a forcing term and \( J \) is the antisymmetric matrix:

\[J = \begin{pmatrix} 0 & I_{\mathbb{R}^n} \\ -I_{\mathbb{R}^n} & 0 \end{pmatrix}.\]

In this case the dual least action principle seems to provide the best results in the simplest way.

2. Autonomous case

In this section we take \( h = 0 \) and we assume that the matrix \( A \) is non symmetric and that the function \( f \) satisfies the assumptions:

(1) \( \forall t \in \mathbb{R}_+^*, \ 0 = f(0) < f(t) \quad \text{and} \quad f'(0) = 0; \)

(2) \( \exists k > 0, \ \exists c \geq 0 : \forall t \in \mathbb{R}_+, \ f(t) \leq \frac{k}{2} t^2 + c; \)

(3) \( \exists \alpha > 0, \ \exists K \in \left[ \frac{k}{2\tau} \left(1 + \|A\|^2\right), +\infty \right] \quad \text{such that} \)

\[\forall t \in [0, \alpha], \ \frac{k}{2} t^2 \leq f(t);\]

where

\[\tau = \sup \left\{ b \cdot (A^* - A) a ; \ a^2 + b^2 = 1 ; \ a, b \in \mathbb{R}^n \right\}\]

and \( A^* \) is the adjoint of \( A \). The corresponding hamiltonian \( H \) is then continuously differentiable and we obtain:

**Theorem 1**

*For all \( T \in \left[ \frac{\pi}{K\tau}, \frac{2\pi}{K(1 + \|A\|^2)} \right] \), the hamiltonian system \((H)\) has a non trivial periodic solution with minimal period \( T \).*
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Proof. We proceed by proving successive lemmas.

We denote by $f^*$ the Legendre transformation of $f$:

$$f^*(s) = \sup \left\{ st - f(t) ; \ t \in \mathbb{R} \right\}.$$  

From the assumptions (2) and (3) we deduce easily the following lemma:

**Lemma 1**

$f^*$ satisfies

\begin{equation}
\forall \ t \in \mathbb{R}, \ f^*(t) \geq \frac{1}{2k} t^2 - c, \tag{4}
\end{equation}

\begin{equation}
\exists r > 0 ; \ \forall |t| \leq r, \ f^*(t) \leq \frac{1}{2K} t^2. \tag{5}
\end{equation}

It is easy to show that the function $H$ is convex and its Legendre transformation $H^*$ is given for $(s,q) \in \mathbb{R}^n \times \mathbb{R}^n$ by:

$$H^*(s,q) = \begin{cases} f^*(|q|) & \text{if } s + A^*q = 0 \\ + \infty & \text{elsewhere}. \end{cases}$$

Now, let $T \in \left[ \frac{\pi}{K\tau}, \frac{2\pi}{K(1+\|A\|^2)} \right]$. We consider the functional $\Phi$ in the space $E_0$ defined by

$$E_0 = \left\{ (-A^*v,v) ; \ v \in L^2(0,T; \mathbb{R}^n), \ \int_0^T v(t) \, dt = 0 \right\},$$

$$\Phi(w) = \frac{1}{2} \int_0^T (Jw, \pi w) \, dt + \int_0^T H^*(w) \, dt,$$

where $\pi w$ is the primitive of $w$ with mean value zero:

$$\frac{d}{dt} (\pi w) = w, \ \int_0^T (\pi w)(t) \, dt = 0.$$

**Lemma 2**

Let, for $w \in L^2(0,T; \mathbb{R}^{2n})$

$$g(w) = \int_0^T H^*(w) \, dt.$$

The subdifferential of $g|_{E_0}$ at a point $w \in E_0$, where $g$ is finite, is given by:

$$\partial g(w) = \left\{ u \in L^2(0,T; \mathbb{R}^{2n}) ; \ \exists x \in E_0^\perp, \ u(t) + x(t) \in \partial H^*(u(t)) \ a.e \right\}$$

where $\partial$ designates the subdifferential in $E_0$. 
Proof. It is clear that
\[ \partial g(w) = \partial (g + \delta_{E_0})(w) \]
where
\[ \delta_{E_0}(w) = \begin{cases} 0 & \text{if } w \in E_0 \\ +\infty & \text{elsewhere.} \end{cases} \]
We have \( \delta_{E_0}^* = \delta_{E_0}^+ \) and
\[ E_0^+ = \mathbb{R}^{2n} + \{ (v, Av); \ v \in L^2(0, T; \mathbb{R}^n) \} . \]
Consequently, for \( u \in L^2(0, T; \mathbb{R}^{2n}) \), we have
\[ (g^* \nabla \delta_{E_0}^*)(u) = \inf_{x \in \mathbb{R}^n} \int_0^T H(u + x) \, dt . \]
So, by the assumption (2), there exist \( \alpha, \beta > 0 \) such that
\[ 0 \leq (g^* \nabla \delta_{E_0}^*)(u) \leq \alpha \| u \|_{L^2}^2 + \beta \]
and, since \( g^* \nabla \delta_{E_0}^* \) is convex, then it is continuous. Noting \( u = (r, p) \), we have
\[ (g^* \nabla \delta_{E_0}^*)(u) = \inf_{\xi \in \mathbb{R}^n} \int_0^T f(\| p - Ar + \xi \|) \, dt . \]
Therefore, by assumptions (2), (3) and the convexity of \( f \), the function \( g^* \nabla \delta_{E_0}^* \) is exactly.

In the other hand, \( g \) and \( \delta_{E_0} \) are convex, l.s.c. and proper. Then for all \( w \in E_0 \), where \( g \) is finite, we have
\[ \partial (g + \delta_{E_0})(w) = \partial g(w) + \partial \delta_{E_0}(w) . \]
The result follows then from the equalities:
\[ \partial \delta_{E_0}(w) = E_0^+ \text{ and } \partial g(w) = \{ u \in L^2(0, T; \mathbb{R}^{2n}); \ u(t) \in \partial H^*(w(t)) \ a.e. \} \]
\[ \square \]

Lemma 3

The function \( \Phi \) has a global minimum on \( E_0 \):
\[ \exists \bar{w} \in E_0, \ \min_{E_0} \Phi = \Phi(\bar{w}) . \]
Proof. By the inequality (4) we have, for all \( w \in E_0; \)
\[
\int_0^T H^*(w) dt \geq \frac{1}{2k(1 + \|A\|^2)} \|w\|^2_{L^2} - cT.
\]
By application of the Wirtinger inequality, we obtain
\[
\Phi(w) \geq \frac{1}{2} \left\{ \frac{1}{k[1 + \|A\|^2]} - \frac{T}{2\pi} \right\} \|w\|^2_{L^2} - cT
\]
and since \( T < \frac{2\pi}{k[1 + \|A\|^2]} \) and the space \( E_0 \) is reflexive the minimum of \( \Phi \) on \( E_0 \) is achieved. \( \square \)

**Lemma 4**

We have \( \min_{E_0} \Phi < 0 \).

Proof. By definition, we have
\[
T > \frac{\pi}{K\tau} = \frac{\pi}{K} \inf \left\{ \frac{a^2 + b^2}{b \cdot (A^* - A)a}, \ b \cdot (A^* - A)a > 0 \right\},
\]
so there exist \( a, b \in \mathbb{R}^n \) such that
\[
b \cdot (A^* - A)a > 0, \ a^2 + b^2 \leq r^2 \quad \text{and} \quad T > \frac{\pi(a^2 + b^2)}{K b \cdot (A^* - A)a}.
\]
Let
\[
v(t) = a \cos \left( \frac{2\pi t}{T} \right) + b \sin \left( \frac{2\pi t}{T} \right), \ w(t) = (- A^*v(t), v(t)),
\]
we have by the inequality (5) and easy calculation
\[
\int_0^T f^*(|v|) dt \leq \frac{1}{2K} \|v\|^2_{L^2} = \frac{a^2 + b^2}{4K},
\]
\[
\int_0^T \langle Jw, \pi w \rangle dt = - \frac{T}{2\pi} b \cdot (A^* - A)a,
\]
consequently
\[
\Phi(w) \leq \frac{1}{4\pi} b \cdot (A^* - A)a \left[ \frac{\pi(a^2 + b^2)}{K b \cdot (A^* - A)a} - T \right] < 0
\]
and so \( \inf_{E_0} \Phi < 0. \)
Now, let \( w \) be a point of \( E_0 \) where the minimum of \( \Phi \) is achieved, then we have

\[
0 \in -J\pi w + \partial g(w).
\]

From Lemma 2, there exist \( \xi \in \mathbb{R}^{2n} \) and \( h \in L^2(0, T; \mathbb{R}^n) \) such that

\[
J(\pi w)(t) + \xi + (h(t), Ah(t)) \in \partial H^*(w(t)) \text{ a.e.}
\]

By setting \( x = J\pi w + \xi \), we obtain by the Fenchel reciprocity \( \dot{x} = JH'(x) \). It is clear that \( x \) is \( T \)-periodic and, by lemma 4, \( x \) is non-constant.

It remains to show that \( T \) is the minimal period of \( x \). Assume that \( x \) and then \( w \) are \( \frac{T}{m} \)-periodic with \( m \geq 2 \). Let \( y(t) = w\left(\frac{t}{m}\right) \), then \((\pi y)(t) = m(\pi m)\left(\frac{t}{m}\right)\).

This point \( y \) belongs to \( E_0 \) and verifies

\[
\Phi(y) = \frac{m}{2} \int_0^T \langle Jw, \pi w \rangle dt + \int_0^T H^*(w) dt \leq m \Phi(w).
\]

So, by lemma 4, we have \( \Phi(y) < \inf_{E_0} \Phi \) which is a contradiction, then \( T \) is the minimal period of \( x \). \( \square \)

### 3. Non autonomous case

Here we assume that \( f \) is not constant and there exist \( k > 0 \) and \( a \geq 0 \) such that:

\[
(6) \quad \forall t \in \mathbb{R}_+, \quad 0 \leq f(t) \leq \frac{k}{2} t^2 + a \quad \text{and} \quad f'(0) = 0,
\]

and the function \( h \) is continuous, periodic with minimal period \( T > 0 \) and mean value zero.

**Theorem 2**

For all \( T \in \left]0, \frac{\pi}{\sqrt{1 + \|A\|^2}} \right] \), the hamiltonian system \((\mathcal{H})\) has a periodic solution with minimal period \( T \).
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Proof. We proceed as in section 2, so we omit some details. We consider the functional $\Phi$ over the space $E$ defined by

$$E = E_0 + h$$

$$\Phi(w) = \frac{1}{2} \int_0^T \langle Jw, \pi w \rangle dt + \int_0^T H^*(t, w) dt$$

where $E_0$ is defined as in section 2.

For $w = (-A^*v, v) + h \in E$, we have

$$\Phi(w) = \frac{1}{2} \int_0^T \langle Jw, \pi w \rangle dt + \int_0^T f^*(|v|) dt.$$  

From the assumption (6), we have

$$\int_0^T f^*(|v|) dt \geq \frac{1}{2k} \|v\|_{L^2}^2 - aT,$$

it follows then, by Hölder inequality, that there exists a constant $c$ such that

$$\Phi(w) \geq \frac{1}{4\pi} \left[ \frac{\pi}{k(1 + \|A\|^2)} - T \right] \|w\|_{L^2}^2 - c,$$

and then the global minimum of $\Phi$ over $E$ is achieved at a point $w$. Therefore, there exists $\xi \in \mathbb{R}^{2n}$ and $r \in L^2(0, T; \mathbb{R}^n)$ such that

$$0 \in -J\pi w + \xi + (r, Ar) + \partial H^*(t, w) \quad a.e.$$  

By using the Fenchel reciprocity, it is clear that the function $x = J\pi w + \xi$ is a periodic solution of $(\mathcal{H})$ with minimal period $T$. □

References