Representation of operators by kernels

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ABSTRACT

We prove that differences of order-continuous operators acting between function spaces can be represented with a pseudo-kernel, proved the underlying measure spaces satisfy certain (rather weak) conditions.

To see that part of these conditions are necessary, we show that the strict localizability of a measure space can be characterized by the existence of a pseudo-kernel for a certain operator.

1. Introduction

The representation of classes of operators between function spaces by kernels is a widely used tool in operator theory and functional analysis with an impressive list of applications. See [1], [3], [4], [7], [8], [10], [11] and [14]. The aim of the present note is to prove a representation theorem for certain operators between function spaces under very weak conditions concerning the underlying measure spaces.

Such a generalization seems to be worthwhile for the following reason: The results available in the literature usually require that one of the measure spaces be Standard Borel, the Borel space of a second countable topological space or something similar which, at least, implies that the Borel σ-algebra is countably generated.

Our Theorem 1.1 shows that such a condition is not necessary.

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In the second section we use the Gelfand isomorphism of \( L_\infty(X, \mathfrak{A}, \mu) \) to show that the assumption of strict localizability of \((X, \mathfrak{A}, \mu)\) in Theorem 1.1 is indispensable. To be more precise, we construct an operator which has a representation with a pseudo-kernel if \( L_\infty(X, \mathfrak{A}, \mu) \) has a lifting.

1. The Representation Theorem

In this section we are going to show that linear operators between spaces of functions which satisfy a certain "continuity property" can be represented by means of pseudo-kernels.

First, we have to recall some measure theoretic notions for which there seems to be no standard terminology. (For detailed information we refer the reader to [5].) Starting with a measure space \((X, \mathfrak{A}, \mu)\), a function \( f : X \to [-\infty, \infty] \) is called \(\mathfrak{A}\)-measurable, if \( \{ f \leq \alpha \} \in \mathfrak{A} \) for every \( \alpha \in \mathbb{R} \) (here \( \{ f \leq \alpha \} = \{ x \in X ; f(x) \leq \alpha \} \)).

We weaken this condition and call \( f : X \to [-\infty, \infty] \) locally \( \mu \)-measurable, if, for every \( A \in \mathfrak{A}_{\text{fin}}^\mu := \{ B \in \mathfrak{A} ; \mu(B) < \infty \} \) there is an \( \mathfrak{A} \)-measurable function which coincides with \( f \) \( \mu \)-a.e. on \( A \) (i.e. on \( A \setminus N \) for some \( N \in \mathfrak{A} \) with \( \mu(N) = 0 \)). \( N \subset X \) is said to be a local \( \mu \)-null set if, for every \( A \in \mathfrak{A}_{\text{fin}}^\mu \) there is an \( N_A \in \mathfrak{A} \) with \( \mu(N_A) = 0 \) such that \( N \cap A \subset N_A \). By \( L_0(X, \mathfrak{A}, \mu) \) we denote the space of all locally \( \mu \)-measurable functions which are finite locally \( \mu \)-a.e. (i.e. outside some local \( \mu \)-null set).

For every \( f \in L_0(X, \mathfrak{A}, \mu) \) its equivalence class is defined as \( \tilde{f} := \{ g \in L_0(X, \mathfrak{A}, \mu) ; f = g \ \text{locally} \ \mu \text{-a.e.} \} \) and the symbol \( L_0(X, \mathfrak{A}, \mu) \) is used for the corresponding space of equivalence classes.

An element \( f \in L_0(X, \mathfrak{A}, \mu) \) is said to be essentially bounded, if \( \| f \|_\infty := \inf \{ c ; |f| \leq c \ \text{locally} \ \mu \text{-a.e.} \} \) is finite (where we use the convention \( \inf \emptyset = \infty \)).

By \( L_\infty(X, \mathfrak{A}, \mu) \) we denote the set of essentially bounded functions in \( L_0(X, \mathfrak{A}, \mu) \), while \( L_\infty(X, \mathfrak{A}, \mu) \) stands for the corresponding space of equivalence classes.

Given a Hausdorff space \( Y \) we say that \( \nu \) is a Radon measure on \( Y \), if \( \nu \) is locally finite (i.e. each point has a \( \nu \)-integrable neighborhood) and \( \nu \) is inner regular, i.e. \( \nu(B) = \sup \{ \nu(K) ; K \subset B, K \text{ compact} \} \) for every \( B \in \mathcal{B}_Y \), where the latter denotes the Borel \( \sigma \)-algebra of \( Y \). We write \( M_R(Y) \) for the space of finite signed Radon measures on \( Y \).

An important condition in our Representation Theorem is the strict localizability of a measure space \((X, \mathfrak{A}, \mu)\). This means that there is a family \( \mathfrak{D} \subset \mathfrak{A}_{\text{fin}}^\mu \) of disjoint sets of nonzero measure which satisfies: \( \forall A \in \mathfrak{A}_{\text{fin}}^\mu, \mu(A) > 0 \ \exists D \in \mathfrak{D} : \mu(A \cap D) > 0 \). Such a family \( \mathfrak{D} \) is called a \( \mu \)-decomposition. We are going to use that every
strictly localizable \((X, \mathfrak{A}, \mu)\) admits a linear lifting of \(L_\infty(X, \mathfrak{A}, \mu)\), i.e. a positive \(\Lambda : L_\infty(X, \mathfrak{A}, \mu) \to L_\infty(E, \mathfrak{A}, \mu)\) with \(\Lambda f \in f\) for all \(f \in L_\infty(X, \mathfrak{A}, \mu)\) and \(\Lambda \mathbb{1} = 1\) (see [5], [6]).

We use \(M_\infty(X, \mathfrak{A}, \mu)\) for the space of equivalence classes of bounded measurable functions.

1.1. Representation Theorem

Let \((X, \mathfrak{A}, \mu)\) be strictly localizable and \(\nu\) be a Radon measure on the Hausdorff space \(Y\) with \(\mathcal{B} = \mathcal{B}_Y\). Let \(T : M_\infty(Y, \mathcal{B}, \nu) \to L_0(X, \mathfrak{A}, \mu)\) be linear and \(T = T_+ - T_-\), with order continuous \(T_+\) and \(T_-\). Then there exists \(\tau : X \to M_R(Y)\) such that

\[
\int f(y)\tau(\cdot)(dy) \in \mathcal{T}\hat{f}
\]

for every bounded, \(\mathcal{B}\)-measurable \(f\).

We express this relation between \(\tau\) and \(T\) by saying that \(\tau\) is a pseudo-kernel for \(T\).

Under more restrictive conditions concerning the underlying measure spaces, similar results have been proved in [1], [3], [4], [7], [8], [11], [12] and [15]. The last reference contains more information concerning the relevant literature and an excellent study of the consequences of theorems of the above kind. We call an operator \(T\) order-continuous, if \(T(\sup F) = \sup T(F)\) for any set \(F\) of functions which is directed under \(\le\). Note that if \(\mu\) is finite (as is the case in [15]), it is sufficient to check this condition for sequences.

1.2. Corollary

With \((X, \mathfrak{A}, \mu)\) and \((Y, \mathcal{B}, \nu)\) as in 1.1, let \(T : L_1(X, \mathfrak{A}, \mu) \to L_1(Y, \mathcal{B}, \nu)\) be linear and bounded. Then there exists a mapping \(\tau : X \to M_R(Y)\) such that

\[
\int_B Tf \, d\nu = \int f(x)\tau(x)(B)\mu(dx)
\]

for every \(f \in L_1(X, \mathfrak{A}, \mu)\) and every \(B \in \mathcal{B}\).

Proof. Apply 1.1 to \(T' : L_\infty(Y, \mathcal{B}, \nu) \to L_\infty(X, \mathfrak{A}, \mu)\) and use that the dual of a positive operator on \(L_1\) is order-continuous. \(\Box\)

Note that in all of the above quoted references the \(\sigma\)-algebra \(\mathcal{B}\) has to be countably generated.

Since in [13] we wanted to apply a result like 1.2 in the case where \(Y\) is an uncountable product, we were forced to prove the above generalization.
Proof of 1.1. Without restriction we may assume $T \geq 0$. Since $T \in L_0(X, A, \mu)$, we find a finite representative $g \in T \in$. The locally $\mu$-measurable sets $X_n := \{g \in [n-1, n]\}$ cover $X$ as $n$ runs through $\mathbb{N}$. Note that $0 \leq Tf \leq n\|f\|_\infty$ on $X_n$ for every $n \in \mathbb{N}$ and $f \in L_\infty(Y, \mathcal{B}, \nu)$.

Let $\Lambda : L_\infty(X, A, \mu) \to \mathcal{L}_\infty(X, A, \mu)$ be a linear lifting. Denote the set of compact subsets of $Y$ by $\mathcal{K}$.

For $K \in \mathcal{K}$ and $\varphi \in C(K)$ let $\varphi^0$ be the function which is 0 on $Y \setminus K$ and equals $\varphi$ on $K$.

Set
\[ \langle \tau(x), \varphi \rangle := \Lambda \left( (T\varphi^0)_{X_n} \right)(x) \]
for $n \in \mathbb{N}, x \in X_n, K \in \mathcal{K}$ and $\varphi \in C(K)$.

Clearly, $\tau(x) \in C(K)'$ for every $x \in X, K \in \mathcal{K}$ and $\tau(x)$ is a measure in the sense of [2; §1, $N^{0.3}$, déf. 5]; see [2; §3, $N^{0.2}$, Thm 2] for the relation with the notion of a Radon measure as given above. Note that we simply write $\tau$ instead of Bourbaki's $\tau^*$.

From the properties of $\Lambda$, we have
\[ \langle \tau(\cdot), \varphi \rangle \in Ts\varphi^0 \]
for all $K \in \mathcal{K}, \varphi \in C(K)$.

It follows that
\[ \tau(\cdot)(U) \in T\chi_U \]
for all open $U \subset Y$.

By its very definition,
\[ \tau(x)(U) = \sup \{ \langle \tau(x), \varphi \rangle; K \in \mathcal{K}, K \subset U, \varphi \in C(\mathcal{K}) \leq \varphi \leq \chi_U \} \]

For $B \subset X_n, \mu(B) < \infty,$
\[ \int_B \tau(x)(U)\mu(dx) = \int_B \sup \langle \tau(x), \varphi \rangle \mu(dx) \]
\[ = \sup \int_B \Lambda \left( (T\varphi^0)_{X_n} \right)(x) \mu(dx) \]
\[ = \sup \int_B T\varphi^0 d\mu = \int_B T\chi_n d\mu , \]
where we used [6; Chap. III, Thm. 3, p. 40] for the crucial second equality sign and the order continuity of $T$ for the last equation.
Now consider $\mathcal{F} := \{ g \text{ bounded, measurable}; \int g(y)\tau(\cdot)(dy) \in T\hat{g} \}$. Then $\chi_U \in \mathcal{F}$ for all open $U \subset Y$ by the last inclusion and the continuity of $T$ implies that $\mathcal{F}$ is closed under monotone limits.

An appeal to a monotone class theorem (eg. [9; App. 1, Lemma 3, p.241]) finishes the proof. □

Some of the ideas in the above proof were inspired by [10].

2. Pseudo–Kernels related to the Gelfand isomorphism

In this section we prove that the use of a lifting in Theorem 1.1 was, in fact, necessary. To be more precise, we show in 2.2 that a certain operator defined with the help of the Gelfand isomorphism of $L_\infty(X, \mathfrak{A}, \mu)$ admits a pseudo-kernel iff $(X, \mathfrak{A}, \mu)$ is strictly localizable.

By its very definition $L_\infty(X, \mathfrak{A}, \mu)$ is isometrically imbedded in $L_1(X, \mathfrak{A}, \mu)'$. From now on we assume that the measure space $(X, \mathfrak{A}, \mu)$ is localizable, i.e. this canonical mapping is surjective. As $L_\infty(X, \mathfrak{A}, \mu)$ is an abelian $C^*$-algebra, there is an isomorphism

$$ \Gamma : L_\infty(X, \mathfrak{A}, \mu) \rightarrow C(\hat{X}) $$

called the Gelfand isomorphism, where $\hat{X}$ denotes the maximal ideal space of $L_\infty(X, \mathfrak{A}, \mu)$.

To simplify notation, we sometimes write $\hat{f}$ instead of $\Gamma f$. For $A \in \mathfrak{A}_{\text{loc}}^\mu := \{ B \subset X : \chi_B \in L_\infty(X, \mathfrak{A}, \mu) \}$ the function $\Gamma \chi_A$ is a continuous idempotent; hence there is a compact open $\hat{A} \subset \hat{X}$ such that

$$ \Gamma \chi_A = \chi_{\hat{A}}. $$

2.1. Lemma

Let $A \in \mathfrak{A}_{\text{fin}}^\mu$ and set

$$ \langle \mu_A, \varphi \rangle := \int_A \Gamma^{-1}\varphi d\mu. $$

Then $\mu_A$ defines a measure on $\hat{X}, \mu_A(X) = \mu(A)$ and the embedding

$$ C(\hat{A}) \hookrightarrow L_\infty(\hat{A}, \mathfrak{B}_{\hat{A}}, \mu_A) $$

is surjective.
Proof. See e.g. [14; Prop. 1.12, p.107]. □

By Zorn’s lemma we find a family \( \mathcal{F} \subset \mathcal{A}_\text{fin}^\mu \) such that \( \mu(E \cap F) = 0 \) for different \( E, F \in \mathcal{F} \), which is maximal with respect to the order given by \( \text{sup} \{ \tilde{\chi}_F; F \in \mathcal{F} \} \). \( L_\infty(X, \mathcal{A}, \mu) \) is order complete as \( (X, \mathcal{A}, \mu) \) is localizable, cf [6; 16.6.4, p.282]).

Let \( X^\# := \bigcup \{ \hat{F}; F \in \mathcal{F} \} \), which is open and dense in \( \hat{X} \). For every \( F \in \mathcal{F} \) the measure \( \hat{\mu}_F \) from 2.1 defines a finite Radon measure on \( X^\# \) with support \( \hat{F} \). Hence

\[
\mu^\# := \sum_{F \in \mathcal{F}} \hat{\mu}_F
\]

defines a Radon measure on \( X^\# \).

Let \( \mathcal{B}^\# \) denote the Borel \( \sigma \)-algebra and set

\[
G : M_\infty(X^\#, \mathcal{B}^\#, \mu^\#) \to L_0(X, \mathcal{A}, \mu)
\]
\[
Gf := \sup_{F \in \mathcal{F}} \Gamma^{-1}(f\chi_{\hat{F}})
\]

which is defined since, by 2.1, \( f\chi_{\hat{F}} \in L_\infty(\hat{F}) \) has a continuous representative.

\[
G\chi_{\hat{A}} = \sup_{F \in \mathcal{F}} \Gamma^{-1}(\chi_{\hat{A} \cap \hat{F}}) = \sup_{F \in \mathcal{F}} (\chi_{\hat{A} \cap F}) = \chi_{\hat{A}}.
\]

Again, the localizability of \( (X, \mathcal{A}, \mu) \) ensures the existence of this supremum.

It remains to check the order continuity of \( G \). To this end it suffices to note that

\[
G_* : L_1(X, \mathcal{A}, \mu) \to L_1(X^\#, \mathcal{B}^\#, \mu^\#),
\]
\[
G_* f := (Tf)^\sim
\]
is a positive contraction with \( (G_*)' \supset G \).

Assume that \( G \) has a pseudo–kernel \( \gamma : X \to M_R(X^\#) \). Changing \( \gamma \) on a local \( \mu \)-null set, if necessary, we may restrict ourselves to the case that \( \gamma(x, X^\#) = 1 \) for all \( x \in X \). For \( f \in L_\infty(X, \mathcal{A}, \mu) \) set

\[
\Lambda f(x) := \langle \gamma(x), \Gamma f \rangle.
\]

Then \( \Lambda \hat{1} = 1 \) by what we just assumed and

\[
\Lambda\tilde{\chi}_A = \chi_A \text{ locally } \mu \text{–a.e.}
\]

for all locally measurable \( A \subset X \).

To prove this last claim, observe that

\[
G\chi_{\hat{A}} = \text{sup} \{ \tilde{\chi}_{A \cap F}; F \in \mathcal{F} \} = \tilde{\chi}_{A},
\]

where we used the maximality of \( \mathcal{F} \) for the last equation.

Linearity and density imply

\[
\Lambda f \in f \text{ for all } f \in L_\infty(X, \mathcal{A}, \mu).
\]

As \( \Lambda \) is clearly positive, it defines a linear lifting. Thus we have proven (iii) \( \Rightarrow (i) \) of the following:
2.2. Theorem

Let \((X, \mathfrak{A}, \mu)\) be complete and localizable. Let \(X^\#, \mu^\#\) and \(G\) be as above. Then the following conditions on \((C, \mathfrak{A}, \mu)\) are equivalent:

(i) \((X, \mathfrak{A}, \mu)\) is strictly localizable.

(ii) The assertion of Theorem 1.1 holds for \((X, \mathfrak{A}, \mu)\).

(iii) The operator \(G : M_\infty(X^\#, \mathfrak{B}^\#, \mu^\#) \to L_\infty(X, \mathfrak{A}, \mu)\) is pseudo-integral.

The other implications are clear.

In [13] we also studied the assumptions (in 1.1) concerning \((Y, \mathfrak{B}, \nu)\) in detail. With arguments similar to those given above we could show the following:

Measure spaces \((Y, \mathfrak{B}, \nu)\) with countably generated \(\mathfrak{B}\) for which the assertion of Theorem 1.1 holds are in an appropriate sense, isomorphic to compact spaces with a Radon measure.

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