SOME RESULTS ABOUT THE SIZE OF THE EXCEPTIONAL SET IN NEVANLINNA'S SECOND FUNDAMENTAL THEOREM

by

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ABSTRACT

Let $F$ be a meromorphic function in the plane. Some conditions are given on the size of the set of positive real numbers, outside which the term $S(r,F)$ which arises in the logarithmic derivative Lemma is small compared with the characteristic function $T(r,F)$ of $F$.

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1. INTRODUCTION

Let $F(z)$ be a meromorphic function in the plane. We shall use the usual notation of Nevanlinna theory. For any complex value $a$ we define

$$m(r,a) = \frac{1}{2\pi} \int_0^{2\pi} \log \left( \frac{1}{|F(re^{i\theta}) - a|} \right) d\theta,$$

$$N(r,a) = \int_0^r t \left( \frac{n(t,a) - n(0,a)}{t} \right) dt + n(0,a) \log r,$$

where $n(t,a)$ denotes the number of roots according with their multiplicities of the equation $F(z) = a$ in $|z| \leq t$.

Similarly we define

$$m(r,\infty) = \frac{1}{2\pi} \int_0^{2\pi} \log + |F(re^{i\theta})| d\theta.$$
\[ N(r, \infty) = \int_0^r \frac{n(t, \infty) - n(0, \infty)}{t} + n(0, \infty) \log r. \]

The function
\[ T(r, F) = m(r, \infty) + N(r, \infty), \]

is called the characteristic function of the meromorphic function \( F \).

Next we state the second fundamental theorem of Nevanlinna.

**Theorem A.**—Let \( F(z) \) be a meromorphic function in the plane. Let \( r \) be a positive real number, \( 0 \leq r < \infty \), and \( a_1, a_2, \ldots, a_q \), are \( q > 2 \) distinct values of the extended complex plane such that \( |a_\mu - a_\nu| \geq \delta \), \( 1 \leq \mu < \nu \leq q \), for a certain \( \delta > 0 \). Then

\[ (q-2) T(r, F) < N(r, a_1) + N(r, a_2) + \ldots + N(r, a_q) - N_1(r) + S(r), \quad (1.1) \]

where \( N_1(r) \) is a positive term given by
\[ N_1(r) = N(r, \frac{1}{F'}) + 2N(r, F) - N(r, F'), \]

and
\[ S(r, F) = m(r, \frac{F'}{F}) + m(r, \sum_{\nu=1}^{q} \frac{F'}{F-a_\nu}) + q \log \frac{3q}{\delta} + \log 2 \]
\[ + \log \frac{1}{F'(0)}, \quad (1.2) \]

with modifications if \( F(0) = \infty \) or \( F'(0) = 0 \).

The quantity \( S(r, F) \) will in general be negligible with respect \( T(r, F) \) and the combination of Theorem A and the estimation for \( S(r, F) \) constitute Nevanlinna’s second fundamental theorem.

The following theorem due to R. Nevanlinna gives an estimation for \( S(r, F) \),

**Theorem B.**—Suppose that \( F \) is a meromorphic function in the plane, \( S(r, F) \) is defined by (1.2) and \( \lambda \) is a positive fixed number, then we have
\[ S(r, F) = O(\log T(r, F)) + O(\log r), \quad (1.3) \]
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as \( r \to \infty \) through all values if \( F(z) \) has finite order and otherwise as \( r \to \infty \) outside a set \( E_\lambda \) satisfying

\[
\int_{E_\lambda} r^\lambda \, dr < \infty. \tag{1.4}
\]

In this paper we show that if we consider, instead of (1.3), the weaker condition,

\[
S(r,F) = o(T(r,F)), \tag{1.5}
\]

we obtain an stronger conclusion than (1.4). We also give a new condition of a different type that (1.4) on the size of the exceptional set, outside which, (1.5) holds. These conditions turn out to be sharp as is proved in [1].

2. STATEMENT OF THE RESULTS

**Theorem 1.**—The error term \( S(r,F) \) in Nevanlinna's second fundamental theorem satisfies (1.5), i.e.

\[
S(r,F) = o(T(r,F)),
\]

as \( r \to \infty \) outside a set \( E \), independent of \( \lambda \), such that

\[
\int_{E} r^\lambda \, dr < \infty \tag{2.1}
\]

for every \( \lambda > 0 \).

**Theorem 2.**—The error term \( S(r,F) \) satisfies (1.5), as \( r \to \infty \) outside a set \( E \), which can be contained in a sequence of intervals \([r_n, r_n + \delta_n]\), such that

\[
\delta_n < \frac{1}{\Psi(n)^2} \quad \text{where} \quad \Psi(1) = 1, \quad \Psi(n) = e^{\Psi(n-1)}. \tag{2.2}
\]

Both conditions (2.1) and (2.2) imply that the exceptional set has finite measure but they are different in character, i.e. neither implies the other for \( \lambda > 0 \).

The condition (2.2) gives a limitation only on the size of the intervals \([r_n, r_n + \delta_n]\) and (2.1) takes into account the position of the exceptional set.
3. SOME PRELIMINARY RESULTS OF THEOREM 1

In the proof of Theorem 1 we shall use the following result which implies in particular Theorem B.

**Theorem C.** Suppose that $F$ is a meromorphic function in the plane and $\Phi(r)$ an increasing function for which there exists a constant $C$ such that

$$\Phi(r+1) \leq C\Phi(r),$$

then

$$S(r,F) \leq 20 \log^+ T(r,F) + 12 \log^+ \Phi(r) + 10 \log^+ r + \text{constant} \quad (3.1)$$

outside a set $E_\Phi$ satisfying

$$\int_{E_\Phi} \Phi(r) \, dr < \infty$$

In the proof of Theorem C we shall use the following lemmas,

**Lemma 3.1. (Logarithmic derivative lemma).** Let $F(z)$ be meromorphic in the plane. For $0 \leq r \leq R$, we have

$$m(r, \frac{F'}{F}) \leq 4 \log^+ T(R,F) + 4 \log^+ \log^+ \frac{1}{|F(0)|} +$$

$$+ 5 \log^+ R + 6 \log^+ \frac{1}{R-r} + \log^+ \frac{1}{r} + 14$$

**Lemma 3.2.** Suppose $T(r)$ continuous, increasing and $T(r) \geq 1$ for $r_0 \leq r < +\infty$ and $\Phi(r)$ increasing for $r_0 \leq r \leq +\infty$ such that

$$\Phi(r+1) \leq C\Phi(r), \quad r \geq r_0$$

for a certain constant $C$. Then we have

$$T \left( r + \frac{1}{\Phi(r)T(r)} \right) < 2T(r),$$
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outside a set \( E_{\Phi} \) satisfying

\[
\int_{E_{\Phi}} \Phi(r) \, dr < \infty
\]

This is Borel's lemma amplified. The proof is the same as the one given in


4. Proof of Theorem C

\( S(r,F) \) was defined in (1.2) and it can be written as

\[
S(r,F) = m(r, \frac{F'}{F}) + m(r, \frac{G'}{G}) + \text{constant,} \quad (4.1)
\]

where

\[
G(z) = \prod_{\nu = 1}^{q} (F(z) - a_{\nu}),
\]

By Lemma 3.1 we have for \( 0 \leq r \leq R \)

\[
m(r, \frac{G'}{G}) \leq 4 \log^+ T(R,G) + 5 \log^+ R + 6 \log^+ \frac{1}{R-r} + \text{constant}
\]

for \( r \) bigger than a certain \( r_0 > 0 \).

We take \( R = r + \frac{1}{\psi(r) T(r,F)} \). Then for \( r \) large we obtain

\[
5 \log^+ R < 5 \log^+ r + \text{constant},
\]

\[
6 \log^+ \frac{1}{R-r} = 6 \log^+ \psi(r) + 6 \log^+ T(r,F)
\]

and by Lemma 3.2 we have

\[
4 \log^+ T(R,G) \leq 4 \log^+ (qT(R,F) + \text{constant}) \leq 4 \log^+ T(r,F) + \text{constant.}
\]

outside a set \( E_{\Phi} \) satisfying

\[
\int_{E_{\Phi}} \Phi(r) \, dr < \infty.
\]

Thus

\[
m(r, \frac{G'}{G}) \leq 10 \log^+ T(r,F) + 6 \log^+ \psi(r) + 5 \log^+ r + \text{constant} \quad (4.2)
\]
and in particular
\[ m(r, \frac{F' \cdot r}{F}) \leq 10 \log^+ T(r,F) + 6 \log^+ \Phi(r) + 5 \log^+ r + \text{constant} \quad (4.3) \]

With (4.1), (4.2) and (4.3) we conclude
\[ S(r,F) \leq 20 \log^+ T(r,f) + 12 \log^+ \Phi(r) + 10 \log^+ r + \text{constant}, \]

which is (3.1).

5. PROOF OF THEOREM 1

We may assume that \( F \) is transcendental, since otherwise there is no exceptional set. Then.

\[ \frac{\log r}{T(r,F)} \to 0 \text{ as } r \to \infty. \]

Let \( a(r) \) be an increasing function such that

\[ a(r) \to \infty, \quad \frac{a(r) \log r}{T(r)} \to 0, \quad r \to \infty \]

\[ (a(r+1) - a(r)) \log r \leq C_1, \quad \frac{a(r)}{r} \leq C_2, \quad \forall r \geq r_0, \]

and set

\[ \Phi(r) = r^{a(r)} \]

Then

\[ \frac{\Phi(r+1)}{\Phi(r)} \leq C \]

so that by Theorem C

\[ S(r,F) \leq 20 \log^+ T(r,F) + 12 \log^+ \Phi(r) + 10 \log^+ r + \text{constant} = \]

\[ = 20 \log^+ T(r,F) + 12 \max(0, a(r) \log r) + 10 \log^+ r + \text{constant} = \]

\[ = O(T(r)), \]
outside a set \( E_\Phi \) with

\[
\int_{E_\Phi} \Phi(r) \, dr < \infty.
\]

Since \( \Phi(r) \geq r^\lambda \), for all \( r \geq r_\lambda \), for every \( \lambda \geq 0 \), we conclude \( \int_{E_\Phi} r^\lambda \, dr < \infty \) and the proof of Theorem 1 is complete.

6. AN AUXILIARY LEMMA TO THEOREM 2

In the proof of Theorem 2 we shall use the following lemma.

**Lemma 6.1.** Suppose that \( T(r) \) is a continuous, increasing real-valued function for \( r_0 \leq r < \infty \), and that \( T(r) \geq 1 \) there. Then we have

\[
T(r + \frac{1}{T(r)}) < e^{T(r)^{1/2}} \tag{6.1}
\]

outside an exceptional set contained in a union of intervals \( \bigcup_n [r_n, r_n + \delta_n] \), such that \( \delta_n \) satisfies

\[ \delta_n < \frac{1}{\Psi(n)^2} \text{ where } \Psi(1) = 1, \Psi(n) = e^{\Psi(n-1)} \text{, i.e. } (2.2) \]

We set \( t(r) = T(r)^{1/2} \), then (6.1) becomes

\[
t(r + \frac{1}{t(r)^2}) < e^{t(r)}. \tag{6.2}
\]

Let \( r_1 \) be the lower bound of all \( r \geq r_0 \) such that (6.2) is false or equivalently the first value for which

\[ t(r_1 + \frac{1}{t(r_1)^2}) \geq e^{t(r_1)}. \]

We write \( r_1' = r_1 + t(r_1)^{-2} \) and let \( r_2 \) be the lower bound of all \( r \geq r_1' \) such that (6.2) is false. We can define in this way a sequence \( r_n \) writing

\[ r_{n-1}' = r_{n-1} + \frac{1}{t(r_{n-1})^2}. \]
and defining $r_n$ as the lower bound of all $r \geq n_1$ such that

$$t(r + \frac{1}{t(r)^2}) \geq e^{t(r)}.$$  

The exceptional set is contained in the union

$$\bigcup_n [r_n, r_n'] = \bigcup_n [r_n, r_n + \frac{1}{t(r_n)^2}].$$

We write $\delta_n = t(r_n)^{-2}$ and since

$$t(r_n) \geq t(r_n') \geq e^{t(r_n-1)} \quad \text{and} \quad t(r_n) \geq 1,$$

we obtain by induction

$$t(r_n) \geq \Psi(n + 1) \geq \Psi(n),$$

and then we conclude

$$\delta_n = \frac{1}{t(r_n)^2} \leq \frac{1}{\Psi(n)^2}.$$  

This completes the proof of Lemma 6.1.

7. PROOF OF THEOREM 2

Again we write $S(r, F)$ in the form

$$S(r, F) = m(r, \frac{F'}{F}) + m(r, \frac{G'}{G}) + \text{constant},$$

where

$$G(z) = \prod_{\nu = 1}^{q} (F(z) - a_\nu).$$

By the logarithmic derivative Lemma, we have for $0 \leq r < R$

$$m(r, \frac{G'}{G}) \leq 4 \log^+ T(R, G) + 5 \log^+ R + 6 \log \frac{1}{R - r} + \log^+ \frac{1}{r} + 14.$$
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Now, we take

\[ R = r + \frac{1}{T(r,F)}, \]

then with the same argument of Theorem C but using Lemma 6.1 instead of Lemma 3.2 we obtain that

\[ S(r,F) \leq 16 T(r,F)^{1/2} + 12 \log^+ T(r,F) + 10 \log^+ r + \text{constant} = \]

\[ = o(T(r,F)), \]

outside a set, which can be contained in a sequence of intervals \([r_n, r_n + \delta_n]\) with \(\delta_n\) satisfying (2.2) and the proof of Theorem 2 is finished.
BIBLIOGRAPHY
