THE APPROXIMATION PROPERTY OF ORDER \( p \)
IN LOCALLY CONVEX SPACES\(^(*)\)

by

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ABSTRACT:

We define the approximation property and the local approximation property of order \( p > 1 \) of a locally convex space. We prove that, if \( E = \lim\limits_{\rightarrow} E_n \) is a regular inductive limit of Frechet spaces \( E_n \) with the approximation property of order \( p > 1 \), then \( E \) has this property.

1. INTRODUCTION

In [12], for every real number \( p \geq 1 \), Saplar defines the approximation property of order \( p \) (in shortly \( AP_p \)) of a Banach space \( E \). The approximation property of order 1 for \( E \) is nothing but the classical approximation property (AP) of Grothendieck. Every Banach space has \( AP_2 \) and every Banach space with the AP has the \( AP_p \) for all \( p > 1 \). In [11], Reinov notices that there are Banach spaces with the \( AP_p \), \( p > 2 \), without the AP and gives an example of a reflexive separable Banach space \( E \) such that, for every \( p \neq 2 \), \( E \) does not have the \( AP_p \).

It seems that there is no definition of the \( AP_p \), \( p > 1 \), for locally convex spaces. The purpose of this paper is to introduce the definition of the approximation property of order \( p > 1 \) of a locally convex space \( E \) and to develop a theory similar to the classical one for \( p = 1 \) as far as possible. This definition is given in section 2. However, after the proof of some general properties, we shall only consider in this paper the problem of the \( AP_p \), \( p > 1 \), of an inductive limit of a sequence of Frechet spaces. Then, we obtain a theorem similar to a result of Bierstedt and Meise ([2]) for the classical approximation property.

Our notation for separated locally convex spaces (in shortly l.c.s.) over the field \( K \) of real numbers \( \mathbb{R} \) or complex numbers \( \mathbb{C} \), is standard and we refer the

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reader to [6] and [13]. Given a l.c.s. E, we shall denote by \( \mathcal{U}(E) \) a basis of absolutely convex closed O-neighbourhoods and by \( \mathcal{K}_U \) (resp. \( \mathcal{K}_U' \)) the canonical map from \( E \) onto \( E_U \) (resp. into \( E'_U \)) for each \( U \in \mathcal{U}(E) \). If \( E \) and \( F \) are l.c.s., \( \mathcal{L}(E,F) \) will be the space of all continuous linear maps from \( E \) into \( F \) and \( \mathcal{B}_k(E'_o,F'_o) \) will be the space of all separately continuous bilinear forms on \( E'_o \times F'_o \) provided with the topology of the uniform convergence on the sets \( U^o \times V^o \), where \( U \in \mathcal{U}(E) \) and \( V \in \mathcal{U}(F) \).

\( \mathbb{N} \) is the set of positive natural numbers. If \( p \in \mathbb{R}, p \geq 1 \), we define its conjugate number \( p' \in [1,\infty] \) such that \( 1/p + 1/p' = 1 \). If \( E \) is a l.c.s. and \( p \geq 1 \), in [1] are defined the spaces \( \mathcal{L}^p(E) \) and \( \mathcal{L}^p[E] \) of weakly \( p \)-summable and absolutely \( p \)-summable sequences \( (x_i) \) of \( E \), respectively. We shall consider \( \mathcal{L}^p(E) \) (resp. \( \mathcal{L}^p[E] \)) endowed with the topology defined by the system of seminorms

\[
\epsilon_{p,U}(x_i) = \begin{cases} 
\sup_{x \in U^o} \left( \sum_{i=1}^{\infty} |x_i|/p \right)^{1/p} & \text{if } 1 \leq p < \infty \\
\sup_{i \in \mathbb{N}} \sup_{x \in U^o} |x_i|x' | & \text{if } p = \infty
\end{cases}
\]

\[
\Pi_{p,U}(x_i) = \begin{cases} 
\left( \sum_{i=1}^{\infty} (p_U(x_i))^{p'/p} \right)^{1/p} & \text{if } 1 \leq p < \infty \\
\sup_{i \in \mathbb{N}} p_U(x_i) & \text{if } p = \infty
\end{cases}
\]

We shall consider every finite sequence \((x_1, x_2, \ldots, x_n)\) as a sequence \((x_i)\) with \( x_i = 0 \) if \( i > n \).

If \( E \) and \( F \) are l.c.s., a map \( T \in \mathcal{L}(E,F) \) is called \( p \)-absolutely summing (\( 1 \leq p < \infty \)) and we will write \( T \in \mathcal{S}^p(E,F) \), if for every \( (x_i) \in \mathcal{L}^p(E) \), we have \((T(x_i)) \in \mathcal{L}^p[F] \). If \( E \) is bornological, the continuity of \( T \) is a consequence of the second condition. The proof is analogous to the proof of [6], pag. 428, in the normed case. A map \( T \in \mathcal{S}^p(E,F) \) such that the map \( \tilde{T}: \mathcal{L}^p(E) \rightarrow \mathcal{L}^p[F] \) defined by means of \( T((x_i)) = (T(x_i)) \), is continuous, will be called totally \( p \)-absolutely summing. If \( E \) is metrizable or \( \mathcal{L}^p(E) \) is quasibarrelled, every \( T \in \mathcal{S}^p(E,F) \) is totally \( p \)-absolutely summing for all l.c.s. \( F \). (The proof is a slight modification of the proof given in [9], pag. 36 in the case \( p = 1 \). It is easy to prove that a map \( T \in \mathcal{L}(E,F) \) is totally \( p \)-absolutely summing if and only if for each \( V \in \mathcal{U}(F) \), there is \( U \in \mathcal{U}(E) \) such that for every \( n \in \mathbb{N} \) and every finite set \( \{x_1, x_2, \ldots, x_n\} \subset E \), we have
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$$\Pi_{p,V}((Tx_i)) \leq \epsilon_{p,U}((x_i)).$$

With the same method of [6] pag. 433, we can prove the following factorization theorem which we shall use later:

**PROPOSITION A:** Let $E,F$ be l.c.s. and $T \in \mathcal{S}(E,F)$ be a totally $p$-absolutely summing map $(1 < p < \infty)$. For each $V \in \mathcal{U}(F)$, there are $U \in \mathcal{U}(E)$, a reflexive Banach space $M$, a totally $p$-absolutely summing map $J_p \in \mathcal{S}(E,M)$ and a map $B \in \mathcal{S}(M,F_V')$, such that

$$K_V T = BJ_p.$$

If $E$ and $F$ are l.c.s. and $p \geq 1$, the topology $\mathcal{g}_p$ of Saphar in the tensor product $E \otimes F$ is defined by the family of seminorms $\{ g_{p,U,V}, U \in \mathcal{U}(E), V \in \mathcal{U}(F) \}$ where

$$g_{p,U,V}(z) = \inf \left\{ \Pi_{p,U}((x_i)) \cdot \epsilon_{p,V}((y_i)) \mid z = \sum_{i=1}^{n} x_i \otimes y_i \in E \otimes F \right\}$$

taking the inf over all representations of $z \in E \otimes F$. With this topology, $E \otimes F$ is denoted by $E \otimes_p F$ and its completion by $E \overset{\otimes_p}{\longrightarrow} F$. In [11] it is proved that, if $E$ and $F$ are Banach spaces, $(E \otimes_p F)' = S^{p'}(F,E')$ where the isomorphism is defined by

$$\langle Tz, z \rangle = \sum_{i=1}^{n} \langle x_i, Ty_i \rangle \forall T \in S^{p'}(F,E'), \forall z = \sum_{i=1}^{n} x_i \otimes y_i \in E \otimes F.$$

It is easy to prove that these property also holds if $F$ is a Frechet space.

In some cases, we shall identify $E \otimes F$ with a subspace of linear mappings from $F'$ into $F$ or from $F'$ into $E$ in the canonical way.

1. **THE APPROXIMATION PROPERTY OF ORDER $p \geq 1$ IN BANACH SPACES**

In what follows, $p$ will be always a real number $p \geq 1$. Let $E,F$ be l.c.s. For each $V \in \mathcal{U}(F), (x_i) \in \mathcal{S}^p(E)$ and $T \in S^p(E,F)$, we define

$$P_{(x_i),V}(T) = \Pi_{p,V}((Tx_i)).$$

We consider $S^p(E,F)$ endowed with the topology $\mathcal{T}_p$, defined by the system of seminorms $\{ P_{(x_i),V}, (x_i) \in \mathcal{S}^p(E), V \in \mathcal{U}(F) \}$. Clearly $\mathcal{T}_p$ is a separated topology.
PROPOSITION 1: If $E$ is a Frechet space and $F$ is a Banach space, the topological dual of $[S^p(E,F), \mathcal{C}_p]$ is a quotient of $F'$ \( \hat{\otimes}_{p'} E \).

Proof. Let $B$ be the closed unit ball of $F$. Every $z \in F' \hat{\otimes}_{p'} E$ has the form (see [4])

$$z = \sum_{i=1}^{\infty} y_i' \otimes x_i$$

with \( y_i' \in \mathcal{P}'(F') \), \( x_i \in \mathcal{P}(E) \).

As $S^p(E,F) \subset (F' \hat{\otimes}_{p'} E)'$, the linear form on $S^p(E,F)$

$$\varphi_z(T) = \langle \varphi_z, T \rangle = \langle z, T \rangle = \sum_{i=1}^{\infty} \langle Tx_i, y_i' \rangle \quad \forall T \in S^p(E,F),$$

is well defined and, by Hölder's inequality

$$|\langle \varphi_z, T \rangle| \leq \Pi_{p',B}(\langle y'_i \rangle) \cdot P_{(x_i),B}(T).$$

Hence \( \varphi_z \in [S^p(E,F), \mathcal{C}_p]' \).

Conversely, let \( \psi \) be an element of \([S^p(E,F), \mathcal{C}_p]'\). Then there is \((x_i) \in \mathcal{P}(E)\) such that

$$|\psi(T)| \leq 1 \quad \text{if } T \in S^p(E,F) \text{ and } P_{(x_i),B}(T) < 1$$

where $B$ is the closed unit ball of $F$. The map $G: [S^p(E,F), \mathcal{C}_p] \rightarrow \mathcal{P}(F)$ such that

$$G(T) = (Tx_i) \quad \forall T \in S^p(E,F)$$

is continuous. Its dual map $G': \mathcal{P}'(F') \rightarrow [S^p(E,F), \mathcal{C}_p]'$ is weakly continuous. (See [7], pag. 359). If $W$ is the closed unit ball of $\mathcal{P}'(F')$, $G'(W)$ is $\sigma ([S^p(E,F), \mathcal{C}_p]', \mathcal{P}(E,F))$-compact and convex. Let us prove that $\psi \in G'(W).$ If not, by [6] pag. 131, there would be $H \in S^p(E,F)$ such that $|\langle H, \psi \rangle| > 1$ and $|\langle H, \eta \rangle| \leq 1$ for all $\eta \in G'(W)$. But, by [7], pág. 196, there is \((y'_i) \in W\) such that

$$P_{(x_i),B}(H) = \langle (Hx_i), (y'_i) \rangle = |\langle G(H), (y'_i) \rangle| = |\langle H, G'((y'_i)) \rangle| \leq 1.$$

Then by hypothesis $|\psi(H)| \leq 1$ which is a contradiction. Then there is \((z'_i) \in W\) such that $\psi = G'((z'_i))$. Now
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$$z = \sum_{i=1}^{\infty} z_i \otimes x_i \in F' \overset{\mathcal{E}}{\otimes} E$$

and it is easy to see that $\langle \varphi_z, T \rangle = \langle T, \psi \rangle$ for every $T \in \mathcal{S}^p(E,F)$. Then $\varphi$ is an epimorphism and $\mathcal{S}^p(E,F)$, \( \mathcal{C}_p \)' $= (F' \overset{\mathcal{E}}{\otimes} E)/\text{Ker}(\varphi)$.

**Proposition 2:** Let $E$ be a Frechet space. The following conditions are equivalent:

1) For every Banach space $F,E' \otimes F$ is $\mathcal{C}_p$-dense in $\mathcal{S}^p(E,F)$.

2) For every Banach space $F$, the canonical map

$$\chi_F : F' \overset{\mathcal{E}}{\otimes} E \rightarrow \mathcal{L}(F,E)$$

is injective.

3) For every Banach space $F$, the canonical map

$$\psi_F : F' \overset{\mathcal{E}}{\otimes} E \rightarrow \mathcal{L}(F,E)$$

is injective.

**Proof:** 1 $\Rightarrow$ 2. Let $\varphi$ be the map from $F'' \overset{\mathcal{E}}{\otimes} E$ onto $[\mathcal{S}^p(E,F')]$, \( \mathcal{C}_p \)' of the proposition 1. Let $z \in F' \overset{\mathcal{E}}{\otimes} E$ be such that $\chi_F(z) = 0$. If

$$z = \sum_{i=1}^{\infty} y_i \otimes x_i$$

with $(y_i) \in \mathcal{E}[F]$ and $(x_i) \in \mathcal{E}^p(E)$, we can consider $z$ as an element of $F'' \overset{\mathcal{E}}{\otimes} E$. Then, for every $y' \in F'$ and every $x' \in E'$ we have

$$0 = \langle \chi_F(z)(y'), x' \rangle = \sum_{i=1}^{\infty} \langle y_i, y' \rangle \langle x_i, x' \rangle = \langle \varphi(z), x' \otimes y' \rangle.$$

Then, since $E' \otimes F'$ is $\mathcal{C}_p$-dense on $\mathcal{S}^p(E,F')$, $\varphi(z) = 0$ on $\mathcal{S}^p(E,F') = (F' \overset{\mathcal{E}}{\otimes} E)'$. Now, for every $T \in \mathcal{S}^p(E,F')$

$$\langle z, T \rangle = \sum_{i=1}^{\infty} \langle Tx_i, y_i \rangle = \langle \varphi(z), T \rangle = 0$$

and hence $z = 0$ and $\chi_F$ is injective.

2) $\Rightarrow$ 3). Given

$$z = \sum_{i=1}^{\infty} y'_i \otimes x_i \in F' \overset{\mathcal{E}}{\otimes} E, \quad (y'_i) \in \mathcal{E}[F'], \quad (x_i) \in \mathcal{E}^p(E),$$
the map $S = \psi_F(z) \in \mathcal{L}(F, E)$ is the restriction to $F$ of the map $X_{F'}(z) \in \mathcal{L}(F'', E)$. Moreover, for each $x' \in E'$

$$g(x') = z' = \sum_{i=1}^{\infty} <x_i, x'> y'_i$$

is a convergent series in $F'$. Given $y'' \in F''$, there is a net $\{y_a, a \in A\}$ in $F$ $\sigma(F'', F')$-convergent to $y''$. Then for every $x' \in E'$

$$|<X_{F'}(z) y'' - X_{F'}(z) y_a, x'>| = |y'' - y_a, g(x')|$$

and

$$X_{F'}(z) y'' = \lim_{a \in A} X_{F'}(z) y_a = \lim_{a \in A} \psi_F(z)(y_a) \quad \text{in} \quad \sigma(E, E').$$

Then, if $z \in F' \hat{\otimes} E$ is such that $\psi_F(z) = 0$, we have $X_{F'}(z) = 0$ in $\mathcal{L}(F'', E)$. By hypothesis $z = 0$ and $\psi_F$ is injective.

3) $\Rightarrow$ 1). Let $G \in [S^{p'}(E, F), \mathcal{C}_{p'}]$, be such that $G(z) = 0$ for every $z \in E' \otimes F$. By proposition 1, there is

$$z = \sum_{i=1}^{\infty} y'_i \otimes x_i \in F' \hat{\otimes} E$$

such that

$$<G, T> = \sum_{i=1}^{\infty} <Tx_i, y'_i> \quad \forall T \in S^{p'}(E, F).$$

Then, for every $x' \in E'$ and every $y \in F$

$$0 = <G, x' \otimes y> = \sum_{i=1}^{\infty} <x', x_i><y, y'_i> = <\psi_F(z)(y), x'>.$$

Hence $\psi_F(z) = 0$ in $\mathcal{L}(F, E)$ and by 3), $z = 0$. Then $G = 0$ and $E' \otimes F$ is $\mathcal{C}_{p'}$-dense in $S^{p'}(E, F)$.

**Proposition 3**: If $E$ is a l.c.s., the following conditions are equivalent:

1) For every Banach space $F$, $E' \otimes F$ is $\mathcal{C}_{p'}$-dense in $S^{p'}(E, F)$.

2) For every l.c.s. $F, E' \otimes F$ is $\mathcal{C}_{p'}$-dense in $S^{p'}(E, F)$.

**Proof**: 1) $\Rightarrow$ 2). Let us suppose that $T_o \in S^{p'}(E, F)$, $(x_i) \in \mathcal{B}^{p'}(E)$ and $V \in \mathcal{U}(F)$. Given $\varepsilon > 0$ we consider the $\mathcal{C}_{p'}$-neighbourhood of $T_o$.
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$$W = \left\{ T \in \mathbb{S}^p(E,F) / P_{\mathcal{V}}(T - T_o) < \epsilon \right\}.$$  

As $\overline{K}_\mathcal{V}T_o \in \mathbb{S}^p(E,H_V)$, by 1) there is

$$T = \sum_{j=1}^{h} x_j' \otimes w_j \in E' \otimes H_V \subset \mathbb{S}^p(E,H_V)$$

such that

$$P_{\mathcal{V}}(\overline{K}_\mathcal{V}T_o - T) < \epsilon/2.$$  

For each $j = 1, 2, \ldots, h$ we have

$$N_j = \sum_{i=1}^{\infty} |<x_j', x_i>|^{p'_1/p'} < \infty.$$  

Let us define $M_j = N_j$ if $N_j \neq 0$ and $M_j = 1$ if $N_j = 0$. Now, we choose $y_j \in F$ such that

$$P_{\mathcal{V}}(w_j - \overline{K}_\mathcal{V}(y_j)) < \epsilon / (2hM_j) \quad \forall j = 1, 2, \ldots, h$$

and we define

$$S = \sum_{j=1}^{h} x_j' \otimes y_j \in E' \otimes F.$$  

Then,

$$P_{\mathcal{V}}(S - T_o) = \left( \sum_{i=1}^{\infty} \left( P_{\mathcal{V}}\left( \sum_{j=1}^{h} |<x_j', x_i>|^{p'_1/p'} \right) \right) + \left( \sum_{i=1}^{\infty} \left( P_{\mathcal{V}}\left( \sum_{j=1}^{h} |<x_j', x_i> - |<\overline{K}_\mathcal{V}(y_j) - w_j, x_i>|^{p'_1/p'} \right) \right) \right.$$
\( \mathcal{L}(F',E) \) in injective. As the projective tensor topology \( \pi \) coincides with the tensor topology \( g_1 \) of Saphar, Reinov ([11]) (and Saphar ([12]) with a slightly different formulation) gave the following definition:

**DEFINITION A**: A Banach space \( E \) has the AP\(_p\) \( (p \geq 1) \) if, for every Banach space \( F \) the canonical map \( x_F \) from \( F \otimes \_p E \) into \( \mathcal{L}(F',E) \) is injective.

Then, the proposition 2 is a new characterization of the AP\(_p\) \( p > 1 \) of a Banach space \( E \).

2. **THE APPROXIMATION PROPERTY OF ORDER \( p > 1 \) IN LOCALLY CONVEX SPACES**

Motivated by propositions 2 and 3, we shall give the following definition: (always \( p > 1 \))

**DEFINITION 1**: A l.c.s. \( E \) is said to satisfy the AP\(_p\), if for every l.c.s. \( F, E' \otimes F \) is \( G_p \)-dense in \( S^p'(E,F) \).

By propositions 3 and 2, this definition is consistent with the definition A of Reinov in the case of Banach spaces \( E \).

**PROPOSITION 4**: Let \( E \) be a l.c.s. with the AP\(_p\). If \( H \) is a dense subspace of \( E \), \( H \) has the AP\(_p\).

**Proof**: Since \( \mathfrak{B}'(H) \subset \mathfrak{B}'(E) \), the proof is immediate.

Consequently, if the completion \( \hat{E} \) of a l.c.s. \( E \) has the AP\(_p\), \( E \) has also the AP\(_p\). Now, we introduce the concept of local approximation property of order \( p \) (local AP\(_p\)):

**DEFINITION 2**: A l.c.s. \( E \) is said to satisfy the local AP\(_p\), if there is a basis of 0-neighbourhoods \( \mathcal{U}_E(E) \) such that the Banach space \( \hat{E} \) has the AP\(_p\) for each \( U \in \mathcal{U}_E(E) \).

In this case, according to proposition 4, each \( E_U \) has the AP\(_p\).

**PROPOSITION 5**: Let \( E \) be a l.c.s. with the local AP\(_p\) and such that, for every l.c.s. \( F \), every \( T \in S^p'(E,F) \) is totally \( p' \)-absolutely summing. Then \( E \) has the AP\(_p\).

**Proof**: Given a l.c.s. \( F, (x_i) \in \mathfrak{B}'(E), V \in \mathcal{U}_E(F) \) and \( T \in S^p'(E,F) \), there is \( U \in \mathcal{U}_E(E) \) such that
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$$\Pi_{p,V}(\{t_j\}) \ll \varepsilon_{p,U}(\{t_j\}) \quad \forall t_1, t_2, \ldots, t_n \in E, \quad \forall n \in \mathbb{N}$$

Then, the map $\overline{T}: E_U \rightarrow \hat{F}_V$ defined by $\overline{T}(\mathcal{K}_U(x)) = (\mathcal{K}_V, T)(x)$ for all $x \in E$, is well defined and $\overline{T} \in \mathcal{S}^p_{p'}(E_U, F_V)$. As $(\mathcal{K}_U(x_j)) \in \mathcal{S}^p_{p'}(E_U)$, by proposition 4, given $\varepsilon > 0$, there is

$$\overline{z} = \sum_{j=1}^{h} x_j \otimes K_V(y_j) \in E_{U'} \otimes F_V$$

such that $P_{\{\mathcal{K}_U(x_j)\}_{V}}(\overline{T} - \overline{z}) \ll \varepsilon$. Then, for

$$z = \sum_{j=1}^{h} x_j \otimes y_j \in E' \otimes F,$$

$P_{\{x_j\}_{V}}(T - z)$ holds and $E$ has the $\text{AP}_p$.

In [9], Nemikovka has shown that in each Frechet space $F$ with a Schauder basis, there is a system $\mathcal{U}_e(F)$ such that for every $U \in \mathcal{U}_e(F)$, $\mathcal{U}_e(F)$ has the $\text{AP}$. Hence, every Frechet space with a Schauder basis has the $\text{AP}_p$ for every $p > 1$.

**THEOREM 1.** Let $E, F$ be l.c.s. such that $F$ has the local $\text{AP}_p$ or $F$ is a Frechet space with the $\text{AP}_p$. Then $E' \otimes F'$ is a $(E \hat{\otimes}_{p} F)'$, $E \hat{\otimes}_{p} F$-dense in $(E \hat{\otimes}_{p} F)'$.

**Proof:** Let us suppose that $F$ is a l.c.s. with the local $\text{AP}_p$ and $z_i, i = 1, 2, \ldots, n$ are in $E \hat{\otimes}_{p} F$. Given $T \in (E \hat{\otimes}_{p} F)'$, there are $U \in \mathcal{U}_e(E)$ and $V \in \mathcal{U}_e(F)$ such that the linear form $T$ on $E_U \hat{\otimes}_{p} F_V$ defined by

$$<(\mathcal{K}_U \hat{\otimes} \mathcal{K}_V)(z), T> = <z, T> \quad \forall z \in E \hat{\otimes}_{p} F,$$

is well defined an $\overline{T} \in (E_U \hat{\otimes}_{p} F_V)' = (\hat{E}_U \hat{\otimes} \hat{F}_V)'$. Let $\varphi \in (E_U \hat{\otimes}_{p} \hat{F}_V)'$ be such that $\varphi (E_U \hat{\otimes}_{p} F_V') = 0$ but $\varphi \neq 0$. By proposition 2, the canonical map $\chi: \hat{E}_U \hat{\otimes} \hat{F}_V \rightarrow (E_U \hat{\otimes}_{p} F_V')$ is injective. Then, there are $x' \in E_{U'}$ and $y' \in F_{V'}$ such that $0 \neq <x(\varphi)(x'), y' > = <\varphi, x' \otimes y'>$, which is a contradiction. Hence $E_{U'} \otimes F_{V'}$ is $\sigma ((\hat{E}_U \hat{\otimes} \hat{F}_V)', (\hat{E}_U \hat{\otimes} \hat{F}_V'))$-dense in $(E_U \hat{\otimes}_{p} F_V)'$.

Now, let $K_U \hat{\otimes} K_V$ be the canonical map from $E \hat{\otimes}_{p} F$ into $E_U \hat{\otimes} F_V = \hat{E}_U \hat{\otimes} \hat{F}_V$. Given $\varepsilon > 0$, there is $w \in E_{U'} \otimes F_{V'} \subset E' \otimes F'$ such that

$$|<\overline{T} - w, (K_U \hat{\otimes} K_V)(z_i)>| = |<T - w, z_i>| \ll \varepsilon \quad i = 1, 2, \ldots, n.$$
and the proof is complete. If \( F \) is a Frechet space with the \( AP_p \), the proof is similar replacing \( F_V \) by \( F \) and using the propositions 3 and 2.

**COROLLARY 1.** Let \( E, F \) be l.c.s. such that \( F \) has the local \( AP_p \) or \( F \) is a Frechet space with the \( AP_p \). Then \( \langle E \hat{\otimes} E', F \hat{\otimes} F' \rangle \) is a dual pair.

**Proof:** Immediate, by theorem 1.

**COROLLARY 2.** Let \( E, F \) be l.c.s. such that \( F \) has the local \( AP_p \) or \( F \) is a Frechet space with the \( AP_p \). Then the canonical map

\[
\hat{\Delta} : E \hat{\otimes} F \rightarrow \hat{\mathcal{B}}_e (E'_a, F'_a)
\]

is injective.

**Proof:** It is easy to see that every \( \varphi \in \hat{\mathcal{B}}_e (E'_a, F'_a) \) can be identified with a bilinear form on \( E' \times F' \). Let \( z \in E \hat{\otimes} F \) be such that \( \hat{\Delta} (z) = 0 \). There is a net \( \{ z_a, a \in A \} \) in \( E \hat{\otimes} F \) convergent to \( z \) in the completion. Then, for every \( x' \in E' \) and \( y' \in F' \),

\[
<z, x' \otimes y'> = \lim_{a \in A} <z_a, x' \otimes y'> = \lim_{a \in A} \hat{\Delta} (z_a) (x', y') = \hat{\Delta} (z) (x', y') = 0.
\]

By theorem 1, \( z = 0 \) and \( \hat{\Delta} \) is injective.

**COROLLARY 3.** Let \( F, G \) be complete l.c.s. such that \( G \) has the local \( AP_p \) or \( G \) is a Frechet space with the \( AP_p \). Let \( H \) be a l.c.s. and \( T \) a continuous injective linear map from \( G \) into \( H \). If \( I \) is the identity map on \( F \), the continuous linear map

\[
I \hat{\otimes} T : F \hat{\otimes} G \rightarrow F \hat{\otimes} H
\]

is injective.

**Proof:** The space \( \mathcal{B}_e (F'_a, G'_a) \) is complete (see [8] pág. 167). We consider the canonical continuous linear maps

\[
\hat{\Delta}_1 : F \hat{\otimes} G \rightarrow \mathcal{B}_e (F'_a, G'_a) \quad \text{and} \quad \hat{\Delta}_2 : F \hat{\otimes} H \rightarrow \mathcal{B}_e (F'_a, H'_a)
\]

as in corollary 2. If \( z \in F \hat{\otimes} G \) is such that \( (I \hat{\otimes} T) (z) = 0 \), we take a net
in $F \otimes G$ convergent to $z$ in $F \otimes G$. Since $\hat{\Delta}_2(I \otimes T)(z) = 0$, given $(x', h') \in F' \times H'$,

$$0 = \lim_{a \in A} |\sum_{i=1}^{n_a} \langle x_i^a, x' \rangle < T y_i^a, h' \rangle| = \lim_{a \in A} |\hat{\Delta}_1 (z_a)(x', T'h')| =$$

$$= |\hat{\Delta}_1 (z)(x', T'h')|.$$

Since $T$ is injective, $T'(H')$ is a $(G', G)$-dense in $G'$. As $\hat{\Delta}_1 (z) \in \mathcal{B}_e (F'_o, G'_o)$, we have $\hat{\Delta}_1 (z) = 0$ on $F' \times G'$. By corollary 2, $z = 0$ and $I \otimes T$ is injective.

3. The approximation property of order $p > 1$ in inductive limits

We begin with a previous result which seems to be interesting in itself.

**Proposition 6:** Let $E$ be a l.c.s. and $(x_i) \in \mathcal{U}^p (E)$. Then there is a bounded set $B$ in $E$ such that $B$ is contained in the closed linear span of $\{ x_i, i \in \mathbb{N} \}$ and $(x_i) \in \mathcal{U}^p (E_B)$.

**Proof:** Set

$$F = \left\{ \sum_{i=1}^{n} b_i x_i \mid b_i \in \mathbb{K}, i = 1, 2, \ldots, n; n \in \mathbb{N} \text{ and } \left( \sum_{i=1}^{n} |b_i|^P \right)^{1/P'} \leq 1 \right\}$$

and

$$\lambda(U) = \sup_{x \in U} \left( \sum_{i=1}^{\infty} |<x_i, x'||^{P} \right)^{1/P} < \infty \quad \forall U \in \mathcal{U}_o (E).$$

By Hölder's inequality, every $z \in F$ lies in $\lambda(U) U^{\infty} = \lambda(U) U$ for each $U \in \mathcal{U}_o (E)$. Then $F$ is bounded and its closed convex hull $B$ is also bounded and is contained in the closed linear span of $\{ x_i, i \in \mathbb{N} \}$.

Let us see that $(x_i) \in \mathcal{U}^p (E_B)$. Let $z'$ be in $(E_B)^{*}$ such that $\| z' \| < 1$ and let $V$ be the closed unit ball of $\mathcal{U}_o^p$. Given $(b_i) \in V$, there is $(c_i) \in \mathbb{K}^\mathbb{N}$ such that $|c_i| = 1$ and $c_i b_i < x_i, z' > = |b_i < x_i, z' >|$ for all $i \in \mathbb{N}$. Then for every $n \in \mathbb{N},$

$$\sum_{i=1}^{n} b_i c_i x_i \in F \subseteq B$$

and

$$\sum_{i=1}^{n} |<x_i, z'|| b_i |\ = \sum_{i=1}^{n} b_i c_i < x_i, z' > = \sum_{i=1}^{n} b_i c_i x_i, z' > \leq \| z' \|.$$
Consequently
\[
\left( \sum_{i=1}^{\infty} |<x_i, z'>|^p \right)^{1/p} = \sup \left\{ \left| \sum_{i=1}^{\infty} b_i <x_i, z'> \right|, (b_i) \in \mathcal{V} \right\} \leq \|z'\|
\]
and \((x_i) \in \ell^p (E_p)\).

In the case \(p = 1\), this result has been obtained by Hollstein in [5].

For our study of the \(\text{AP}_p\) on inductive limits, we shall need the following lemmas:

**LEMMA 1**: Let \(E\) be a Frechet space and \((x_i) \in \ell^p' (E)\). The closed absolutely convex cover of the set
\[
K = \left\{ \sum_{i=1}^{\infty} b_i x_i / b_i \in \mathbb{K}, i = 1, 2, \ldots, t; t \in \mathbb{N} \text{ and } \left( \sum_{i=1}^{\infty} |b_i|^p \right)^{1/p} \leq 1 \right\}
\]
is a \((E, E')\)-compact.

**Proof**: By the theorems of Krein an Eberlein ([7], pág. 325 and 313), it is enough to see that each sequence
\[
z_j = \sum_{i=1}^{\infty} b_i x_i, \quad b_i^j = 0 \text{ if } i > t_j, \quad j \in \mathbb{N},
\]
in \(K\) has a \((E, E')\)-convergent subsequence.

For every \(j \in \mathbb{N}\), \((b_i^j)_{i=1}^{\infty}\) belongs to the closed unit ball \(B\) of \(\ell^p\). Then there is a subsequence (again denoted by \((b_i^j)\)) weakly convergent in \(\ell^p\) to a sequence \((b_i) \in B\). Since \((x_i) \in \ell^p' (E)\), it is easy to see that
\[
z = \sum_{i=1}^{\infty} b_i x_i \in E.
\]
Then, given \(x' \in E'\), the sequence \(<x_i, x'>_{i=1}^{\infty} \in \ell^{p'}\) and hence, \(z = \lim_{j \to \infty} z_j\) in \((E, E')\). This completes the proof.

**LEMMA 2**: Let \(E\) be a reflexive Banach space, \(F\) a l.c.s. and \((x_i) \in \ell^p' (F)\). Let \(B\) be the closed unit ball of \(\ell^p\). Then, the set
\[
H = \left\{ \sum_{i=1}^{\infty} a_i f_i \otimes x_i / \|f_i\| \leq 1 \quad \forall i \in \mathbb{N} \text{ and } (a_i) \in B \right\},
\]
is contained in \(E \hat{\otimes}_p F\) and is \((E \hat{\otimes}_p F, (E \hat{\otimes}_p F')\)-relatively compact.
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\[ \sum_{i=1}^{\infty} a_i f_i \otimes x_i, \quad \| f_i \| \leq 1 \quad \forall i \in \mathbb{N}, \ (a_i) \in B \]

is convergent in \( E \hat{\otimes} F \). As this space is complete, it will be also complete for the finer topology \( r(E \hat{\otimes} F, (E \hat{\otimes} F)') \). By Eberlein’s theorem ([7], pag. 313) the lemma will be proved if we show that each sequence

\[ z_n = \sum_{i=1}^{\infty} a_i f_i^n \otimes x_i \quad n \in \mathbb{N}, \]

in \( H \) has a weakly convergent subnet. Let \( U \) be the closed unit ball of \( E \) endowed with the induced topology by \( \sigma(E, E') \). We consider on \( B \) the induced topology by \( \sigma(q^p, q^{p'}) \). Then the topological space

\[ X = B \times U \times U \times U \times U \ldots \]

equipped with the product topology, is compact. Given the sequence on \( X \)

\[ w_n = ((a_i^n), f_1^n, f_2^n, \ldots, f_j^n, \ldots), \quad n \in \mathbb{N}, \]

there is a subnet \( \{ w_{n(d)} \}, d \in D \) such that

\[ \lim_{d \in D} (a_i^n(d)) = (a_i) \in B \text{ in } \sigma(q^p, q^{p'}) \quad (1) \]

and

\[ \lim_{d \in D} f_j^n(d) = f_j \in U \text{ in } \sigma(E, E') \quad \forall j \in \mathbb{N}. \quad (2) \]

Let us see that

\[ \sum_{i=1}^{\infty} a_i f_i \otimes x_i = \lim_{d \in D} z_n(d) \]

in \( E \hat{\otimes} F, (E \hat{\otimes} F)' \). Let \( \varphi \in (E \hat{\otimes} F)' \). There is \( V \in \mathcal{U}, (F) \) such that the linear form

\[ < \varphi, \sum_{i=1}^{n} x_i \otimes K_V(y_i) >= < \varphi, \sum_{i=1}^{n} x_i \otimes y_i >, \quad \forall \sum_{i=1}^{n} x_i \otimes K_V(y_i) \in E \otimes F_V \]
is well defined and \( \varphi \in SP' (\bar{F}_V, E') \). As \((\bar{K}_V(x_i)) \epsilon \mathcal{P}' (\bar{F}_V)\), we have 
\( \| \varphi(\bar{K}_V(x_i)) \| \epsilon \mathcal{P}' \). Given \( \epsilon > 0 \), by (1), (2) and the inequalities of Hölder and Minkowski, there are \( r \epsilon \mathbb{N} \), \( t \epsilon \mathbb{N} \) and \( d_0 \epsilon D \) such that if \( d \geq d_0 \),

\[
\begin{align*}
| < \sum_{i=1}^{\infty} (a_i^n d_i f_i^n) - a_i f_i, \varphi | & \leq \\
& < \sum_{i=1}^{t} (a_i^n d_i f_i^n) - a_i f_i, \varphi(\bar{K}_V(x_i)) > + \\
& + \sum_{i=t+1}^{\infty} (a_i^n d_i f_i^n) - a_i f_i, \varphi(\bar{K}_V(x_i)) > + \\
& + \sum_{i=t+1}^{\infty} < a_i (f_i^n - f_i), \varphi(\bar{K}_V(x_i)) > + \\
& + \sum_{i=t+1}^{\infty} < a_i (f_i^n - f_i), \varphi(\bar{K}_V(x_i)) > \leq \epsilon
\end{align*}
\]

and the proof is complete.

**Lemma 3:** Let \( M \) be a reflexive Banach space and let \( E = \lim E_n \) be an inductive limit of Frechet spaces \( E_n \) such that each \( E_n \) has the AP. Then, for every \( n \epsilon \mathbb{N} \), \( E' \otimes M \) is \( \mathcal{T}_p \)-dense in \( E' \otimes M \).

**Proof:** We fix \( n \epsilon \mathbb{N} \). If \( \mathcal{I} \) is the identity map on \( M', \mathcal{I} \) is its dual identity map on \( M' \) and \( I_n \) is the inclusion of \( E_n \) into \( E \), by corollary 3, the canonical map

\[ \mathcal{I} \hat{\otimes} I_n : M' \hat{\otimes} E_n \rightarrow M' \hat{\otimes} E \]

is injective. We define

\[ H = \left\{ z \epsilon M' \hat{\otimes} E / < z, M \hat{\otimes} E' > = 0 \right\} = (M \otimes E')_1 \text{ in } M' \hat{\otimes} E, \]

and we consider the canonical quotient map \( K_H \) from \( M' \hat{\otimes} E \) onto the quotient space \( N = (M' \hat{\otimes} E) / H \). Each \( K_H(z) \epsilon N \) defines an element \( \varphi_z \) of the algebraic dual \( (M \otimes E')^* \) by means of \( < \varphi_z, u > = < z, u > \) for all \( u \epsilon M \otimes E' \). By the definition of \( H \), if \( K_H(z) = K_H(w) \), we have \( \varphi_z = \varphi_w \). Moreover, the map \( D : K_H(z) \rightarrow \mathbb{F}_z \) is injective because \( \varphi_z = 0 \) implies \( z \epsilon H \), that is, \( K_H(z) = 0 \).

Let us see that \( J = D K_H(\mathcal{I} \hat{\otimes} I_n) \) is also injective. Let us suppose that \( J(z) = 0 \). Then \( (\mathcal{I} \hat{\otimes} I_n)(z) \epsilon H \) and for every \( m \epsilon M \) and every \( x' \epsilon E' \) we have

\[ 0 = < (\mathcal{I} \hat{\otimes} I_n)(z), m \otimes x' > = < z, (\mathcal{I} \hat{\otimes} I_n)(m \otimes x') > = < z, m \otimes I_n(x') > = (I_n) \]
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Now, given $m \in M, y' \in E_n'$ and

$$w = \sum_{j=1}^{\infty} m_j' \otimes e_j \in M' \hat{\otimes}_{sp} E_n \text{ with } (m_j')_e \in \mathcal{M}' \text{ and } (e_j) \in \mathcal{M}'(E_n)$$  \hspace{1cm} (2)

(see [4]), since $I_n'(E')$ is $o(E_n', E_n)$-dense in $E_n'$ (and hence $\tau(E_n', E_n)$-dense), by lemma 1, there is a net $\{x'_a, a \in A\}$ in $E'$ such that, given $\epsilon > 0$,

$$\sup \left\{ \left| \frac{1}{\sum_{j=1}^{t} bj e_j', y' - I_n'(x'_a)} \right| \left/ b_j \right| \leq K \right. \text{ for some } k \in \mathbb{N}, \text{ by } (2), \text{ we have}$$

$$\sum_{j=1}^{h} \left| m_j', m \right| \left| e_j, y' - I_n'(x'_a) \right| = \left| \sum_{j=1}^{h} \left| m_j', m \right| \left| e_j, y' - I_n'(x'_a) \right| \leq$$

$$\leq \epsilon \left( 1 + \sum_{j=1}^{\infty} \left| m_j', m \right| \right)^{1/p}.$$  \hspace{1cm} (3)

This proves that $m \otimes y' = \lim_{a \in A} m \otimes I_n'(x'_a)$ in $o((M' \hat{\otimes}_{sp} E_n)', (M' \hat{\otimes}_{sp} E_n))$. Then, by (1), $\langle z, M \otimes E_n' \rangle = 0$. By theorem 1, $z = 0$ and $J$ is injective.

It is easy to see that $J$ is continuous from $M' \hat{\otimes}_{sp} E_n$ into $(M \otimes E')^*$ when this space is endowed with the topology $o((M \otimes E')^*, M \otimes E')$. Then, $J'(M \otimes E')$ is $o((M' \hat{\otimes}_{sp} E_n)', (M' \hat{\otimes}_{sp} E_n))$-dense in $(M' \hat{\otimes}_{sp} E_n)'$ and also is $\tau((M' \hat{\otimes}_{sp} E_n)', \ M' \hat{\otimes}_{sp} E_n)$-dense.

We consider now

$$z = \sum_{r=1}^{k} x_r' \otimes m_r \in E_n' \otimes M, \quad z' = \sum_{r=1}^{k} m_r \otimes x_r' \in M \otimes E_n,'$$

$(x_r') \in \mathcal{M}'(E_n)$ and $\epsilon > 0$. By lemma 2, the set

$$P = \left\{ \sum_{i=1}^{\infty} a_i f_i' \otimes x_i / a_i \in \mathbb{K} \ \forall i \in \mathbb{N}; \left( \sum_{i=1}^{\infty} \left| a_i \right|^p \right)^{1/p} \leq 1; \left\| f_i' \right\| \leq 1 \ \forall i \in \mathbb{N} \right\}$$

is $o((M' \hat{\otimes}_{sp} E_n), (M' \hat{\otimes}_{sp} E_n))$-relatively compact. Then, there are
\[ w = \sum_{h=1}^{\infty} u_h \otimes y_h \in E' \otimes M \quad \text{and} \quad w^I = \sum_{h=1}^{\infty} y_h \otimes u_h^I \in M \otimes E' \]
such that

\[
\sup_{v \in P} |\langle z^I - J' (w^I), v \rangle| = \sup_{v \in P} |\langle z^I, v \rangle - \langle J(v), w^I \rangle| = \sup_{v \in P} |\langle z^I - w^I, v \rangle| \leq \epsilon \tag{3}
\]

Now, we choose, for every \( i \in \mathbb{N} \), an element \( \tilde{r}_i \in M' \) such that \( \| \tilde{r}_i \| \leq 1 \) and

\[ \| (z^I - w^I) (x_i) \| = \langle (z^I - w^I) (x_i), \tilde{r}_i \rangle = \langle z^I - w^I, \tilde{r}_i \otimes x_i \rangle. \]

Then, by (3)

\[
\left( \sum_{i=1}^{\infty} \| (z^I - w^I) (x_i) \|^p 1/p' \right)^{1/p} = \sup \left\{ \| \sum_{i=1}^{\infty} a_i \tilde{r}_i \| (z^I - w^I) (x_i) \| / \left( \sum_{i=1}^{\infty} |a_i|^p 1/p \leq 1 \right) \right\} \leq \epsilon
\]

and the lemma is proved with help of \( w \).

**Theorem 2:** Let \( E = \lim \rightarrow E_n \) be a regular inductive limit of Fréchet spaces \( E_n \) such that every \( E_n \) has the AP_p. Then \( E \) has the AP_p.

**Proof:** Given a l.c.s. \( F, \varphi \in \mathcal{S}^p' (E,F), (x_i) \in \mathcal{S}^p' (E), V \in \mathcal{F}_E (F) \) and \( \epsilon > 0 \), we must show that there is \( w \in E' \otimes F \) such that \( P_{(x_i), V} (\varphi - w) \ll \epsilon \).

By proposition 6, there is a bounded set \( B \) in \( E \) such that \( (x_i) \in \mathcal{S}^p' (E) \). Since \( E \) is regular, there is \( n \in \mathbb{N} \) such that \( B \subset E_n \subset E_n \) and \( B \) is bounded in \( E_n \). Then \( (x_i) \in \mathcal{S}^p' (E_n) \) and the restriction \( \varphi_n \) of \( \varphi \) to \( E_n \) belongs to \( \mathcal{S}^p' (E_n,F) \). By proposition A, there are a reflexive Banach space \( M \), a map \( A \in \mathcal{S}^p' (E_n,M) \) and a map \( B_0 \in \mathcal{S}' (M, \tilde{F}_V) \) such that \( \tilde{F}_V \varphi_n = B_0 A \). Moreover, we can suppose that \( A(E_n) = M \) restricting \( B_0 \) to the reflexive Banach space \( \overline{\mathcal{F}_E (E_n)} \) if necessary.

Let \( \| B_0 \| \) be the norm of the map \( B_0 \). Since \( E_n \) has the AP_p, there is \( z \in E_n' \otimes M \) such that

\[
\left( \sum_{i=1}^{\infty} \| A(x_i) - z (x_i) \|^p 1/p' \right)^{1/p} \ll \epsilon/3 (1 + \| B_0 \|).
\]
By lemma 3, there is
\[ t = \sum_{j=1}^{k} x_j' \otimes m_j \in E' \otimes M \]
such that
\[ (\sum_{i=1}^{\infty} \| (z-t)(x_i) \|^p)^{1/p'} \leq \varepsilon/3 (1 + \| B_0 \|). \tag{1} \]
If we define
\[ \eta = 1 + \sum_{j=1}^{k} (\sum_{i=1}^{\infty} |<x_j', x_i>|^p)^{1/p'}, \tag{2} \]
since $A(E_n)$ is dense in $M$, we choose $e_j \in E_n, j = 1, 2, \ldots, k$, such that
\[ \| A(e_j) - m_j \| \leq \varepsilon/3 \eta (1 + \| B_0 \|) \quad \forall j = 1, 2, \ldots, k. \tag{3} \]
Then, if
\[ w = \sum_{j=1}^{k} x_j' \otimes \varphi_n(e_j) \in E' \otimes F, \]
we have
\[ P(x_j, V)(\varphi - w) = P(x_j, V)(K_V \varphi_n - K_V w) = \]
\[ = (\sum_{i=1}^{\infty} (p_V(B_0 A)(x_i) - \sum_{j=1}^{k} |<x_j', x_i>| B_0 A(e_j))^{p'})^{1/p'} \leq \]
\[ \leq \| B_0 \| \left( \sum_{i=1}^{\infty} \| A(x_i) - \sum_{j=1}^{k} |<x_j', x_i>| A(e_j) \|^{p'})^{1/p'} \leq \right. \]
\[ \leq \| B_0 \| \left( \sum_{i=1}^{\infty} \| A(x_i) - z(x_i) \|^p \right)^{1/p'} + \| B_0 \| \left( \sum_{i=1}^{\infty} \| z(x_i) - t(x_i) \|^p \right)^{1/p'} + \]
\[ + \| B_0 \| \left( \sum_{i=1}^{\infty} \sum_{j=1}^{k} |<x_j', x_i>| (m_j - A(e_j)) \|^{p'} \right)^{1/p'} \leq \varepsilon \]
by Minkowski's inequality and (1), (2) and (3). Then, the proof is complete.
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