ON INCLUSION RELATIONS BETWEEN LORENTZ SEQUENCE SPACES AND INEQUALITIES OF LEWIS TYPE

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ABSTRACT:

Are well known the Lorentz sequence spaces \( \ell_{p,q} \) and the relations
\( \ell_{p,q} \subset \ell_{p_1,q_1} \) if \( 1 \leq p < p_1 \leq \infty \) and \( \ell_{p,q} \subset \ell_{p,q_1} \) if \( 1 \leq p < \infty, 1 \leq q < q_1 \leq \infty \) [4], [5], [1]. In this paper a generalization of these spaces is considered using the symmetric norming functions of \( K \). Schatten [2], [8], [9] and some inclusion relations are presented. In the second part of the paper these inclusion relations are utilised to establish inequalities of Lewis type [1], [7] for some operator ideals generated by an additive s-function (s-number) [5], [6], [11].

1. INCLUSION RELATIONS BETWEEN THE SPACES \( \ell_{p,q} \)

Let be \( C_0 \) the space of all real sequences converging to 0 and \( \hat{C} \) a subspace of \( C_0 \) containing the sequences of finite rank, i.e \( x \in \hat{C} \) if \( x = (x_1, x_2, \ldots, x_{\eta}, 0, \ldots) \), \( \eta < \infty \).

A function \( \varphi : \hat{C} \to K \) is called symmetric norming function (\( \varphi \in \mathcal{N} \)) [2], [8], [9] if the following properties are verified.

1. \( \varphi \) is a norm on \( \hat{C} \).
2. \( \varphi(1,0,0,\ldots) = 1 \)
3. \( \varphi((x_1, x_2, \ldots, x_{\eta}, 0, \ldots) = \phi((\|x_{i_1}\|, \|x_{i_2}\|, \ldots, \|x_{i_{\eta}}\|, 0, \ldots)) \), where \( i_1, i_2, \ldots, i_{\eta} \) is a permutation of the set \( 1, 2, \ldots, \eta \).

If \( x \in C_0 \) but \( x \notin \hat{C} \) then \( \varphi(x) = \sup_{\eta} \varphi(x_1, x_2, \ldots, x_{\eta}, 0, \ldots) \).

In the sequel we consider only the sequences \( x \in C_0 \) i.e. \( x_0 \in C_0 \) and \( x_i \geq 0 \) for all \( i \). We consider also that the sequences are written in nonincreasing rearrangement.
We denote $\phi(\eta) = \phi(1, 1, \ldots, 1, 0, \ldots)$ and $\mathcal{N}^* = \left\{ \phi \in \mathcal{N} : \phi(\eta) < \frac{\phi(\eta + 1)}{1-n} \right\}$, $n = 1, 2, \ldots$

Let be $x \in C_0$, $\phi \in \mathcal{N}$ and $\psi \in \mathcal{N}^*$, $1 \leq p, q \leq \infty$. We say that $x \in \mathfrak{K}_{\phi, \psi, p, q}$ if $\|x\|_{\phi, \psi, p, q} = \left\{ \phi(\psi(\eta)^\frac{1}{p} x_{\eta}) \right\}^{\frac{1}{q}} < \infty (\|x\|_{\phi, \psi, p, q}^0 = \phi_{\omega} (\|\psi(\eta)^\frac{1}{p} x_{\eta}\|)^*)^1$.

In a simple way it results that $\mathfrak{K}_{\phi, \psi, p, q}$ is a quasinormed vector space. For the particular case when $\phi(x) = \psi(x) = \phi_1(x) = \sum |x_i|$ it results the spaces $\mathfrak{K}_{p, q}$.

For $\phi = \psi = \phi_1$ we have the spaces $\mathfrak{K}_{p, q}$, $\phi \in \mathcal{N}^*$. ($\mathfrak{K}_{\phi, p, q} = \left\{ x_i \in C_0 : \sup \phi(\eta) x_{\eta} < \infty \right\}$).

From the properties of the functions $\phi$, it results

**Lemma 1**: For all $\phi \in \mathcal{N}^*$ and $\alpha \in [1, \infty)$, $\eta = 1, 2, \ldots$ the following relation holds

$$\phi(\eta)^\alpha - \phi(\eta - 1)^\alpha \leq \alpha \phi(\eta)^{\alpha - 1}.$$  

**Proof**: $\phi(\eta) \leq \phi(\eta - 1) + \phi(1) = \phi(\eta - 1) + 1$

Hence

$$\phi(\eta)^\alpha - \phi(\eta - 1)^\alpha \leq \phi(\eta)^{\alpha} - [\phi(\eta) - 1]^\alpha \leq \alpha \cdot \phi(\eta)^{\alpha}.$$  

**Remark.** For $\alpha \in (0, 1)$ it results $\phi(\eta)^\alpha - \phi(\eta - 1) \geq \alpha \phi(\eta)^{\alpha - 1}(\eta)$.

**Lemma 2**: If $1 \leq i \leq n$ then there exists $c$, $0 < c < 1$, such that

$$c \cdot \phi^\alpha(n) \leq \psi(\phi^\alpha - \phi^\alpha(i - 1)), \text{ for all } \phi \in \mathcal{N}^*, \alpha < \infty, \psi \sim \phi_1.$$  

The proof is easy and we omit it.

Recall also that, for all $\phi \in \mathcal{N}$, $\phi(x) \leq \phi(y)$ if $x_\eta \leq y_\eta$, $\eta = 1, 2, \ldots, [2]$.

**Lemma 3**: Let $1 \leq p, q < \infty$ and $x \in \mathfrak{K}_{\phi, \psi, p, q}$. Then for every $i$ holds

1) $x_i \leq c(p, q, \psi(\phi) ) \cdot \phi(i)^\frac{1}{p} \cdot \|x\|_{\psi, p, q}$ if $1 \leq p < \infty, \phi \sim \phi_1$.

2) $x_i \leq \psi(\phi) \cdot \phi(i)^\frac{1}{p} \cdot \|x\|_{\psi, p, q}$ if $1 \leq q < \infty$ and $\phi(\eta) \geq \psi(\eta), \eta = 1, 2, \ldots$

*) $\phi_{\omega} (\{x_\eta\}) = \sup_{\eta} x_\eta [2]$. 


Proof. If \( x \in \ell_{\psi, \psi, p, q} \), it results
\[
\|x\|_{\ell_{\psi, \psi, p, q}} \geq \phi \left( \psi(K)^{\frac{q}{p}} \cdot x_k \right) \geq \phi \left( \psi(K)^{\frac{q}{p}} \right)
\]
From Lemma 1 it follows
\[
\|x\|_{\ell_{\psi, \psi, p, q}} \geq \frac{1}{p} x_k \cdot \phi \left( \psi(K)^{\frac{q}{p}} \right)
\]
By Lemma 2 we obtain
\[
\|x\|_{\ell_{\psi, \psi, p, q}} \geq c \frac{1}{p} x_k \cdot \psi(i)^{\frac{q}{p}}
\]
Hence \( x_k \leq C_1 \frac{1}{p} \cdot \psi(i)^{\frac{q}{p}} \cdot \|x\|_{\ell_{\psi, \psi, p, q}} \) if \( 1 \leq p < q \).

In case of \( 1 \leq q < p < \infty \), \( \left\{ \psi(K)^{\frac{q}{p}} \right\} \) is nonincreasing and therefore
\[
\|x\|_{\ell_{\psi, \psi, p, q}} \geq x_k \cdot \psi(i)^{\frac{q}{p}} \cdot \phi(i) \geq x_k \cdot \psi(i)^{\frac{q}{p}}.
\]
Hence \( x_k \leq \psi(i)^{\frac{1}{p}} \cdot \|x\|_{\ell_{\psi, \psi, p, q}} \).

We can prove now the inclusion relation

**Proposition 4.** 1) Let \( 1 \leq p < \infty, 1 \leq q < q_1 < \infty \). Then \( \ell_{\psi, \psi, p, q} \subset \ell_{\psi, \psi, q_1, q} \) for every \( \psi \in \mathcal{N}_p \), \( \psi \in \mathcal{N}^* \) and for every \( x \in \ell_{\psi, \psi, p, q} \) \( \|x\|_{\ell_{\psi, \psi, p, q}} \leq K \left( \frac{q_1}{p} \right)^{\frac{1}{q_1}} \) if \( p > q \), \( \phi \sim \psi \) if \( p < q \).

2) Let either \( 1 \leq p < p_1 < \infty, 1 \leq q < \infty \) or \( 1 \leq p < p_1 < q_1, q = \infty \). Then \( \ell_{\psi, \psi, p, q} \subset \ell_{\psi, \psi, q_1, q} \) and for every \( x \in \ell_{\psi, \psi, p, q} \) \( \|x\|_{\ell_{\psi, \psi, q_1, q}} \leq \|x\|_{\ell_{\psi, \psi, p, q}} \). (*)

**Proof.** Let \( x \in \ell_{\psi, \psi, p, q} \) and \( 1 \leq p < q < q_1 < \infty \). Then by using Lemma 3 we have
\[
\|x\|_{\ell_{\psi, \psi, q_1, q}} = \phi \left( \left\{ \psi(i)^{\frac{q_1}{p}} \cdot x_k \right\} \right) = \phi \left( \left\{ \psi(i)^{\frac{q_1}{p}} \cdot x_k \right\} \right) = \phi \left( \left\{ \psi(i)^{\frac{q_1}{p}} \cdot x_k \right\} \right) = \left( \frac{q_1 - q}{p} \right) x_k \cdot \psi(i)^{\frac{q_1 - q}{p}} \cdot \|x\|_{\ell_{\psi, \psi, p, q}}.
\]

(*) In (1) if \( q < p \) it is necessary that \( \phi \) \( \sim \psi \) for \( \eta = 1, 2, \ldots \).
\[
= c \left( \frac{q}{p} \right)^{q_1 - q} \| x \|^q_{\mathcal{L}_{\psi, p, q}} \cdot \phi \left( \left\{ \psi(i)^{p-1} x^{q_1 - q} \right\} \right) = c \left( \frac{q}{p} \right)^{q_1 - q} \| x \|^q_{\mathcal{L}_{\psi, p, q}}
\]

Whence we obtain the first part of (1) \((K = e^{q_1})\).

In case of \(p \gg q\) we deduce

\[
\| x \|^q_{\mathcal{L}_{\psi, p, q_1}} = \phi \left( \frac{q_1 - q}{p} \right) x^{q_1 - q} x^{q_1} = \phi \left( \left\{ \psi(i) \right\}^{p-1} x^{q_1 - q} x^{q_1} \right) \leq
\]

\[
= \| x \|^q_{\mathcal{L}_{\psi, p, q}} \cdot \phi \left( \left\{ \psi(i) \right\}^{p-1} x^{q} \right) \leq \| x \|^q_{\mathcal{L}_{\psi, p, q}}.
\]

For \(1 \leq p < q < q_1 = \infty\) and \(1 \leq q < p < q_1 = \infty\) the relations result directly from Lemma 3. The proof of (2) is easy using the monotonicity of the functions \(\phi \in \mathcal{N}\).

**Remark.** These relations are valid for the spaces \(\mathcal{L}_{\psi, p, q}\) for all \(\psi \in \mathcal{N}\) if \(p > q\) since \(\psi(\eta) = \psi(\eta)\) and for \(\phi \sim \phi_n\) if \(p < \psi\). For \(\phi_n\) see [2].

**Problem.** If \(\psi(\eta) > \psi(\eta), \eta = 1, 2, \ldots\) is the relation \(\mathcal{L}_{\psi, p, q} \subset \mathcal{L}_{\psi, p, q_1}\) true for \(q < p, q < q_1\)?

2. Inequalities of Lewis type

By means of the spaces \(\mathcal{L}_{\psi, p, q}\) we define some operator ideals. Let \(E, F\) be normed spaces and \(\mathcal{L}(E, F)\) the set of all linear and bounded operators \(T: E \to F\). The \(n^{th}\) approximation number of \(T\) is defined to be \(a_n(T) = \inf K \in \mathcal{K} \| T - K \|\), \(K \in \mathcal{L}(E, F)\), rank \(K < n, n = 1, 2, \ldots\). For the definition of others \(s\)-numbers of \(T\) we refer to [5], [6], [7], [9]. It is well known that the approximation numbers are additive and multiplicative i.e.

\[
a_{m + n - 1}(T_1 + T_2) \leq a_m(T_1) + a_n(T_2) ; \quad a_{m + n - 1}(T_1 T_2) \leq a_m(T_1) a_n(T_2).
\]

**\(*)\) From the definition of \(\mathcal{L}_{\psi, p, q}\) it results that \(\{ a_\eta(T) \} \in \mathcal{C}_0\) hence \(T\) is an approximable operator.
On inclusion relations between Lorentz sequence spaces

We denote by \( L_{\phi,p,q}^{(a)}(E,F) = \{ T \in \mathcal{L}(E,F) : \{ a_n(T) \} \in \mathcal{L}_{\phi,p,q} \}^{**} \). This class \( L_{\phi,p,q}^{(a)}(E,F) \) is an operator ideal (cf. [5], [7], [9], [11]) and \( L_{\phi,p,q}^{(a)}(T) = \left\{ \phi \left( \left\{ \phi(n)\frac{p}{q} - 1 \right\} a_n(T) \right\} \right\}^{1 \frac{1}{q}} \) is a quasinorm on \( L_{\phi,p,q}^{(a)}(E,F) \).

Let \( [\mathfrak{L}_1, \mathcal{A}_1] \) and \( [\mathfrak{L}_2, \mathcal{A}_2] \) be quasinormed operator ideals. If there are the constants \( \lambda \gg 0, c \gg 1 \) such that the quasinorms verify \( A_1(S) \leq c \cdot n^{\lambda} A_2(S) \) for \( S \in \mathcal{L}(E,F) \), rank \( S \leq n \), \( n = 1, 2, \ldots \), then this inequality is called inequality of Lewis type [1], [5].

In the sequel we present a generalized form of this inequality for the operator ideals \( L_{\phi,p,q}^{(a)}(E,F) \), using the results from Proposition 4.

Thus we obtain

**Proposition 5.** Let \( 1 \leq p < q \leq \infty, 1 \leq u, v < \infty \), then

\[
L_{\phi,p,u}^{(a)}(S) \leq c \cdot \phi(\eta)^{\frac{1}{p}} \cdot L_{\phi,q,v}^{(a)}(S) \quad \text{for} \quad S \in \mathcal{L}(E,F), \text{rank} \ S \leq n, \ n = 1, 2, \ldots
\]

and \( \phi \in \mathcal{N}^* \), \( \phi \sim \phi_u \).

**Proof.** Put \( \frac{1}{r} = \frac{1}{p} - \frac{1}{q} \). In case \( 1 \leq v < \infty \) we have

\[
L_{\phi,p,u}^{(a)}(S) = \left\{ \phi \left( \left\{ \phi(i)^{\frac{\mu}{r} - 1} a_i(S)^{\mu} \right\}_{i=1}^\infty \right) \right\}^{1 \frac{1}{\mu}} = \left\{ \phi \left( \left\{ \phi(i)^{\frac{\mu}{r} - 1} + \frac{\mu}{r} a_i(S)^{\mu} \right\}_{i=1}^\infty \right) \right\}^{1 \frac{1}{\mu}} \leq \left\{ \phi \left( \left\{ \phi(i)^{\frac{\mu}{r} - 1} \right\}_{i=1}^\infty \right) \right\}^{1 \frac{1}{\mu}} \cdot L_{\phi,q,v}^{(a)}(S) \leq c \cdot \phi(\eta)^{\frac{1}{r}} L_{\phi,q,v}^{(a)}(S).
\]

**Remarks.**

1) This inequality is also valid for any other additive s-number (for example the Gelfand and Kolmogorov numbers [5], [6], [9]).

2) Since \( \phi(\eta) \leq \phi_1(\eta) = \eta \) it results that \( L_{\phi,p,u}^{(a)}(S) \leq c \cdot \eta^{\frac{1}{r}} L_{\phi,q,v}^{(a)}(S) \).
REFERENCES


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