A NOTE ON THE LIFTING OF LINEAR AND LOCALLY CONVEX TOPOLOGIES ON A QUOTIENT SPACE

by

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Abstract. Let \((X, \mathcal{E})\) be a Hausdorff locally convex space, let \(L \subset X\) be a closed linear subspace, and let \(\mathcal{G}\) be a Hausdorff locally convex topology on the quotient space \(X/L\) which is coarser than the quotient topology \(\mathcal{E}/L\). We prove that there exists a Hausdorff locally convex topology \(\mathcal{Z}\) on \(X\) which is coarser than \(\mathcal{E}\) such that the corresponding quotient topology \(\mathcal{Z}/L\) coincides with \(\mathcal{G}\). This proves a statement of G. Köthe [6].

The above conclusion may fail if \((X, \mathcal{E})\) is a topological vector space which is not necessarily locally convex. Moreover, even if \((X, \mathcal{E})\) is a Banach space, \(\mathcal{Z}\) cannot be chosen such that, in addition, the relative topologies \(\mathcal{Z}/L\) and \(\mathcal{E}/L\) induced on \(L\) coincide.

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Notations. For a topological vector space \((X, \mathcal{E})\) and a linear subspace \(L \subset X\), let \(\mathcal{E}/L\) denote the relative topology induced by \(\mathcal{E}\) on \(L\) and let \(\mathcal{Z}/L\) denote the quotient topology on the quotient space \(X/L\). By \(\mathcal{N}_0(X, \mathcal{E})\) we denote the filter of all \(O\)-neighbourhoods in \((X, \mathcal{E})\) and by \((X, \mathcal{E})'\) (or \(X'\)) its topological dual. Given a dual pair \((X, Y)\), let \(\sigma(X, Y)\) and \(\tau(X, Y)\) denote the weak topology and the Mackey topology on \(X\), respectively. For a subset \(A \subset X\), let \(A^0:=\{y \in Y : |\langle x, y \rangle| \leq 1\ \text{for all} \ x \in A\}^\circ\).
Introduction. Let $L$ be a linear subspace of a topological vector space $(X, \mathcal{V})$.

(1) If $\mathcal{S}$ is a linear topology on $L$ which is coarser than $\mathcal{V}|L$ then the triple $((X, \mathcal{V}), L, \mathcal{S})$ has the following extension property, which is well known and can be verified immediately:

The set $\{U + V : U \in \mathcal{V}(X), V \in \mathcal{V}(L, \mathcal{S})\}$ is a $O$-nbhd-basis for a linear topology $\hat{\mathcal{V}}$ on $X$ which satisfies $\hat{\mathcal{V}} \subset \mathcal{V}$, $\hat{\mathcal{V}}|L = \mathcal{V}|L$, and $\hat{\mathcal{V}}|L = \mathcal{S}$ (in fact, $\hat{\mathcal{V}}$ is the finest linear topology on $X$ with these three properties). Thus, if $\mathcal{S}$ is Hausdorff and $L$ is closed in $(X, \mathcal{V})$, then also $\hat{\mathcal{V}}$ is Hausdorff. Moreover, $\hat{\mathcal{V}}$ is locally convex, whenever $\mathcal{V}$ and $\mathcal{S}$ are locally convex.

On the other hand, if $\mathcal{S}$ is supposed to be finer than $\mathcal{V}|L$, an analogous extension property does not hold, as the following example shows:

Let $(X, \mathcal{V})$ by a Mackey space such that $(L, \mathcal{V}|L)$ is not a Mackey space. Assume that there exists a locally convex topology $\hat{\mathcal{V}}$ on $X$ such that $\hat{\mathcal{V}} \subset \mathcal{V}$, $\hat{\mathcal{V}}|L = \mathcal{V}|L$, and such that $\hat{\mathcal{V}}|L$ coincides with the Mackey topology $\tau(L, (L, \mathcal{V}|L))$ where $\tau(L, (L, \mathcal{V}|L))$ satisfies the hypotheses of [2; Lemma 1] (i.e., they are comparable and induce the same topologies on $L$ and on $X/L$) and hence coincide. Since $\mathcal{V}$ is a Mackey topology, we obtain that $\hat{\mathcal{V}} \subset \mathcal{V}$, which is a contradiction.

(2) If $\mathcal{S}$ is a linear topology on $X/L$ which is finer than $\mathcal{V}/L$, then the triple $((X, \mathcal{V}), X/L, \mathcal{S})$ has the following lifting property, the proof of which is again routine and is therefore omitted:

The set $\{u \cap q^{-1}(V) : u \in \mathcal{V}(X), V \in \mathcal{V}(X/L, \mathcal{S})\}$ (where $q : X \to X/L$ denotes the quotient map) is a $O$-nbhd-basis for a linear topology $\check{\mathcal{V}}$ on $X$ which satisfies $\check{\mathcal{V}} \subset \mathcal{V}$, $\check{\mathcal{V}}|L = \mathcal{V}|L$, and $\check{\mathcal{V}}|L = \mathcal{S}$ (in fact, $\check{\mathcal{V}}$ is the coarsest topology on $X$ with these three properties). $\check{\mathcal{V}}$ is locally convex, whenever $\mathcal{V}$ and $\mathcal{S}$ are locally convex.

Finally, suppose that $\mathcal{S}$ is coarser than $\mathcal{V}/L$. The aim of this note is to investigate the lifting properties of such a triple $((X, \mathcal{V}), X/L, \mathcal{S})$. 


Main results. The following proposition has been stated by G. Köthe in [6; p. 190]; however, the proof contained a gap.

Proposition. Let \((X, \mathfrak{F})\) be a Hausdorff locally convex space over the field \(K \in (\mathbb{R}, \mathbb{C})\), let \(L \subset X\) be a closed linear subspace and let \(\mathfrak{E}\) be a locally convex topology on \(X/L\) which is coarser than \(\mathfrak{F}/L\). Then there exists a locally convex topology \(\mathfrak{F}\) on \(X\) with the following properties: \(\mathfrak{F} \subset \mathfrak{F}^L\), \(\mathfrak{F}/L = \mathfrak{E}\), and \(\mathfrak{F}/L = \sigma(L, L')\) (where \(L' = (L, \mathfrak{F}/L)\)).

In particular, \(\mathfrak{F}\) is Hausdorff if \(\mathfrak{E}\) is Hausdorff.

Proof. Let \(M \subset X'\) be a linear subspace such that \(L^0 \cap M = \{0\}\) and \(L^0 + M = X'\). Then the initial topology \(\mathfrak{F}\) on \(X\) w.r. to the quotient map \(\tilde{q} : X \to (X/L, \mathfrak{E})\) and all linear forms \(f : X \to K\) with \(f \in M\), is locally convex and coarser than \(\mathfrak{F}\). Moreover, \(\mathfrak{F}/L = \sigma(L, M) = \sigma(L, L')\) by G. Köthe [7; p. 275, (1) a] and \(q : (X, \mathfrak{F}) \to (X/L, \mathfrak{E})\) is continuous. Thus it remains to prove that \(q : (X, \mathfrak{F}) \to (X/L, \mathfrak{E})\) is open. Let \(U \in \mathfrak{U}_0(X, \mathfrak{F})\). Then there are \(V \in \mathfrak{U}_0(X/L, \mathfrak{E})\) and \(f_1, \ldots, f_n \in M\) such that \(U \supset q^{-1}(V) \cap \bigcap_{1 \leq i \leq n} \ker f_i\), whence \(q(U) \supset V \cap q\left(\bigcap_{1 \leq i \leq n} \ker f_i\right)\).

We show that \(N := L + \bigcap_{1 \leq i \leq n} \ker f_i\) is equal to \(X\). In fact, let \(f \in X'\) satisfy \(N \subset \ker f\). Then \(f \in L^0\). On the other hand, \(f\) belongs to the linear span of \(\{f_1, \ldots, f_n\}\) by Kelley, Namioka [5; p. 7, 1.3]. Thus \(f \in M \cap L^0 = \{0\}\). Since \(N\) is closed in \((X, \mathfrak{F})\), we obtain that \(N = X\).

Now \(q\left(\bigcap_{1 \leq i \leq n} \ker f_i\right) = q(N) = X/L\), whence \(q(U) \supset V\) and consequently \(q(U) \in \mathfrak{U}_0(X/L, \mathfrak{E})\).

Remark. Since there are many choices of \(M\) in \(X'\), the topology \(\mathfrak{F}\) in the above proposition is not unique. Moreover, \(L^0\) may have two algebraically complementary spaces \(M_1, M_2\) in \(X'\) such that \(M_1 + M_2 = X'\). In this case, the supremum of the corresponding two topologies \(\tilde{\mathfrak{F}}_1\) and \(\tilde{\mathfrak{F}}_2\) is finer than \(\sigma(X, X')\) and hence may fail to induce the topology \(\mathfrak{E}\) on \(X/L\). Consequently, if \((X, \mathfrak{F}), L, \mathfrak{E}\) are given as in the above proposition, there is i.e. no finest (linear or locally convex) topology \(\tilde{\mathfrak{F}}\) on \(X\) satisfying \(\tilde{\mathfrak{F}} \subset \mathfrak{F}\) and \(\tilde{\mathfrak{F}}/L = \mathfrak{E}\).

This last statement had been observed by V. Eberhardt already in 1972.
Moreover, in the above proposition \( \hat{x}|L = \sigma(L, L') \) cannot be replaced by \( \hat{x}|L = x|L \), as the following example shows.

**Example 1.** Let \((X, \mathcal{Z})\) be a Hausdorff locally convex space containing a linear subspace \(L\) such that \((L, \mathcal{Z}|L)\) is a Banach space and such that \(L\) is not topologically complemented in \((X, \mathcal{Z})\). (Thus \((X, \mathcal{Z})\) may be any Banach space which is not a Hilbert space). Let \(\mathcal{E}\) be a weak topology on \(X/L\) which is coarser than \(\mathcal{Z}/L\). Then there is no locally convex topology \(\mathcal{Z}\) on \(X\) such that \(\mathcal{Z} \subset \mathcal{Z}\), \(\mathcal{Z}|L = = \mathcal{Z}|L\), and \(\mathcal{Z}|L = \mathcal{E}\).

**Proof.** Let us assume that such a topology \(\mathcal{Z}\) exists. Then there exists an absolutely convex \(O\)-neighbourhood \(U\) in \((X, \mathcal{Z})\) such that \(U \cap L\) is a bounded \(O\)-neighbourhood in \((L, \mathcal{Z}|L) = (L, \mathcal{Z}/L)\). The seminormed topology \(\mathcal{Z}_U\) generated by the Minkowski functional \(p_U\) on \(X\) clearly satisfies \(\mathcal{Z}_U \subset \mathcal{Z} \subset \mathcal{Z}\) and \(\mathcal{Z}_U|L = \mathcal{Z}|L\).

Since \(\mathcal{E}\) is a weak topology, there is an \(\mathcal{E}\)-closed linear subspace \(M \subset X/L\) of finite codimension such that \(q(U) \supset M\) (where \(q: X \to X/L\) denotes the quotient map). Let \(N := q^{-1}(M)\). Since \(q(U \cap N) = \epsilon q(U \cap q^{-1}(M)) = \epsilon q(U) \cap M\) for every \(\epsilon > 0\), we obtain that \((\mathcal{Z}_U|N)/L\) equals the coarsest topology on \(N/L\). Consequently, \(L\) is dense in \((N, \mathcal{Z}_U|N)\). As \(\mathcal{Z}_U|L = \mathcal{Z}|L\) is a Banach space topology, we obtain that \((N, \mathcal{Z}_U|N)\) is the topologically direct sum of \(L\) and the \((\mathcal{Z}_U|N)\)-closure \(P\) of \((0)\). Hence, because of \(\mathcal{Z}|N \supset \mathcal{Z}_U|N\) and \(\mathcal{Z}|L = \mathcal{Z}_U|L\), also \((N, \mathcal{Z}|N) = L \oplus P\). Now finally, \(N\) being closed and of finite codimension in \((X, \mathcal{Z})\), we get that \(L\) is topologically complemented in \((X, \mathcal{Z})\), a contradiction.

A slight modification of the above proof shows that there is not even a locally pseudoconvex topology \(\mathcal{Z}\) on \(X\) satisfying \(\mathcal{Z} \subset \mathcal{Z}\), \(\mathcal{Z}|L = \mathcal{Z}|L\), and \(\mathcal{Z}|L = \mathcal{E}\).

A Hausdorff topological vector space \((X, \mathcal{Z})\) is called minimal, if there is no Hausdorff linear topology on \(X\) which is strictly coarser than \(\mathcal{Z}\). It is an open problem, whether all Hausdorff quotients of a minimal topological vector space are again minimal. If an analogue to the above proposition for linear (instead of locally convex) topologies \(\mathcal{Z}\) and \(\mathcal{E}\) were valid, one would get a positive solution of this problem immediately. The following example will show that there is no hope to solve the problem in this way.
A note on the lifting of linear and locally convex topologies

Let us first recall some well-known facts about the so-called associated locally convex topologies which we will use in Example 2. For a topological vector space \((Z, \mathfrak{B})\) let \(\mathfrak{B}_k\) denote the finest locally convex topology on \(Z\) which is coarser than \(\mathfrak{B}\). Then \((\mathfrak{B}/Y)_k = \mathfrak{B}_k/Y\) for every linear subspace \(Y \subset Z\), \((\mathfrak{B}/Y)_k = \mathfrak{B}_k/Y\) for every dense linear subspace \(Y \subset (Z, \mathfrak{B})\), and \(\langle \cdot, \cdot \rangle\) commutes with the formation of finite products.

**Example 2.** Let \(\mathfrak{B}\) denote the product topology on \(\omega := K^N\), where \(K \in (\mathbb{R}, \mathfrak{C})\). Since \(\omega\) has the same dimension as the space of all measurable functions \(f: [0, 1] \to K\), there exists a metrizable linear topology \(\mathfrak{D}\) on \(\omega\) satisfying \((\omega, \mathfrak{D})' = (0)\). The linear infimum \(\mathfrak{D} \land \mathfrak{B}\) is a weak topology on \(\omega\) without nontrivial continuous linear forms, hence equal to the coarsest topology \(\emptyset\) on \(\omega\). Thus, by \([1; Lemma 1(a)]\), the diagonal \(\Delta := \{(x, x) \in \omega \times \omega : x \in \omega\}\) is a dense linear subspace of the product space \((\omega, \mathfrak{D}) \times (\omega, \mathfrak{B})\).

Next we choose a non-zero element \(a \in \omega\) and define the one-dimensional linear subspace \(L := K a \times (0) \subset \omega \times \omega\).

Then \(X := L + \Delta\) provided with the relative topology \(\mathfrak{F}\) induced by the product topology \(\mathfrak{D} \times \mathfrak{B}\) is a metrizable topological vector space, which is dense in \((\omega, \mathfrak{D}) \times (\omega, \mathfrak{B})\). Hence \(\mathfrak{F}_k = (\mathfrak{D} \times \mathfrak{B})_k/X = (\emptyset \times \mathfrak{B})_k/X\). From this we obtain that \(\mathfrak{F}_k/L\) is the coarsest topology on \(L\); moreover, again by \([1; Lemma 1(a)]\), \(X/L\) provided with the locally convex topology \(\mathfrak{E} := \mathfrak{F}_k/L = (\mathfrak{F}/L)_k\) is topologically isomorphic to \((\omega, \mathfrak{B})\).

We show that there is no Hausdorff linear topology \(\tilde{\mathfrak{F}}\) on \(X\) satisfying \(\tilde{\mathfrak{F}} \subset \mathfrak{F}\) and \(\tilde{\mathfrak{F}}/L = \emptyset\).

In fact, it follows from Kalton, Peck \([4; 3.4\ and\ 3.5]\) that for any such topology \(\tilde{\mathfrak{F}}\) the space \((X, \tilde{\mathfrak{F}})\) would be topologically isomorphic to \((\omega, \mathfrak{B})\) (see Drewnowski \([3; p. 99,\ section\ 4.]\)). Hence \(\tilde{\mathfrak{F}}\) would be locally convex and thus coarser than \(\mathfrak{F}_k\). But this is impossible since \(\mathfrak{F}_k\) is not Hausdorff.

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REFERENCES


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