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VECTOR VALUED MEASURES OF BOUNDED MEAN OSCILLATION

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0. Introduction

The duality between H^1 and BMO, the space of functions of bounded mean oscillation (see [**JN**]), was first proved by C. Fefferman (see [**F**], [**FS**]) and then other proofs of it were obtained. Using the atomic decomposition approach ([**C**], [**L**]) the author studied the problem of characterizing the dual space of H^1 of *vector-valued* functions. In [**B2**] the author showed, for the case $\Omega = \{|z| = 1\}$, that the expected duality result H^1-BMO holds in the vector valued setting if and only if X^* has the Radon-Nikodym property. If we want to get a duality result valid for all Banach spaces we may consider vector valued measures (see [**BT**], where the vector valued L_p case is treated, for an explanation) and therefore to deal with the general case it was necessary to consider a new space of vector valued measures closely related to BMO (see [**B1**]).

In this paper we shall study such space in little more detail and we shall consider the H^1-BMO duality for vector-valued functions in the more general setting of spaces of homogeneous type (see [**CW**]).

Throughout the paper X will stand for a Banach space, Ω will be a space of homogeneous type (see definition in the preliminary section) and we write $L_p(\Omega, X)$ for the space of measurable functions on Ω with values in X such that $\|f(x)\|$ belongs to $L_p(\Omega)$. As usual C will denote a constant not necessarily the same at each occurrence.

1. Preliminaries

A space of homogeneous type Ω is a topological space endowed with a Borel measure m and a quasi-distance d, that is $d : X \times X \to \mathbb{R}^+$ with

a) $$d(x, y) = d(y, x),$$

b) $$d(x, y) = 0 \qquad \text{if and only if} \qquad x = y,$$

c) $$d(x, y) \leq K(d(x, z) + d(z, y)).$$

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and we assume that the balls $B_r(x) = \{y \in \Omega : d(x,y) < r\}$ form a basis of open neighborhoods of the point x and there exists a constant A satisfying

$$(1.0) \qquad m(B_r(x)) \leq A\, m(B_{r/2}(x))$$

From (1.0) we can assume that $0 < m(B) < \infty$ for every ball B (otherwise m would be identically 0 or ∞) and therefore m is a σ-finite measure on Ω. Denote by Σ_0 the ring of bounded measurable sets. The σ-finiteness condition implies that the σ-algebra generated by Σ_0 coincides with the Borel σ-algebra that we shall denote by Σ.

Let us now recall the notion of atom with values in X. Given $1 < p \leq \infty$, a function a in $L_p(\Omega, X)$ is called (X,p)-*atom* if

a) the support is contained in a ball $B = B_r(x_o)$

b) $$\left(\frac{1}{m(B)} \int_B \|a(x)\|^p\, dm(x)\right)^{1/p} \leq \frac{1}{m(B)} \quad (p < \infty)$$

b') $$\|a(x)\| \leq \frac{1}{m(B)} \qquad m - a.e. \qquad (p = \infty)$$

c) $$\int_B a(x)\, dm(x) = 0$$

In the case $m(\Omega) < \infty$ the constant function $\frac{1}{m(\Omega)}\, b$, where $b \in X$ with $\|b\| = 1$, is also considered as a (X,p)-atom.

Note that the atoms are in the unit ball of $L_1(\Omega, X)$.

Following **[CW]** we define $H^1_p(\Omega, X)$ as the space of functions f in $L_1(\Omega, X)$ admitting an atomic decomposition

$$(1.1) \qquad f = \sum_{j=0}^{\infty} \lambda_j a_j$$

where the a_j's are (X,p)-atoms and $\sum_{j=0}^{\infty} |\lambda_j| < \infty$. (The convergence of (1.1) is taken in $L_1(\Omega, X)$).

We get a Banach space if we consider the norm

$$\|f\|_{H^1_p} = \inf \sum_{j=0}^{\infty} |\lambda_j|$$

where the infimum is taken over all representations $f = \sum_{j=0}^{\infty} \lambda_j a_j$.

The same arguments as in [**CW**] show that, in fact, for $1 < p, r \leq \infty$

$$H^1_p(\Omega, X) = H^1_r(\Omega, X) \text{ (with equivalent norms)} . \tag{1.2}$$

Let us also recall the definition of vector-valued BMO. Let $1 \leq q < \infty$, an X-valued function which is locally in $L_q(\Omega, X)$ is said to belong to $BMO_q(\Omega, X)$ provided that

$$\sup_{ball B} \Big(\frac{1}{m(B)} \int_B \|g(x) - g_B\|^q dm(x)\Big)^{1/q} \leq C \tag{1.3}$$

where $g_B = \frac{1}{m(B)} \int_B g(x)\, dm(x)$.

Let us denote by

$$\|g\|_{*,q} = \sup\{\Big(\frac{1}{m(B)} \int_B \|g(x) - g_B\|^q\, dm(x)\Big)^{1/q} : \text{ B ball}\}$$

When $m(\Omega) = \infty$ then $\|g\|_{BMO_q} = \|g\|_{*,q}$ gives a norm on the set of equivalence classes of functions which differ by a constant in X.

For $m(\Omega) < \infty$ we consider the norm $\|g\|_{BMO_q} = \|g\|_{*,q} + \| \int_\Omega g(x)\, dm(x)\|$.

Let us recall now a few definitions about vector-valued measures we shall use later on. Let (Ω, Σ, m) be any σ-finite measure space, A a measurable set and $1 < p < \infty$. Given a vector valued measure G, we denote by $|G|$ the variation of G, that is

$$|G|(A) = \sup\{\sum_{i=1}^{n} \|G(E_i)\| : (E_i) \text{ partition of A}\} \tag{1.4}$$

and by $|G|_p(A)$ the p-variation on A, that is

$$|G|_p(A) = \sup\{\Big(\sum_{i=1}^{n} \frac{\|G(E_i)\|^p}{m(E_i)^{p-1}}\Big)^{1/p}\} \tag{1.5}$$

where the supremum is taken over all finite partitions (E_i) of disjoint measurables sets contained in A with $m(E_i) > 0$.

For the case $p = \infty$ we shall denote by $V^\infty(\Omega, X)$ the space of X-valued measures G satisfying

$$\|G(E)\| \leq C\, m(E) \text{ for all measurable set E} \tag{1.6}$$

Defining the norm by the infimum of the constants satisfying (1.6) we get a Banach space.

Remark 1.1. It is not hard to see that in fact $\|G(E_i)\|$ can be replaced by $|G|(E_i)$ in the definition of p-variation. (See Lemma 1 in [**B3**])

Remark 1.2. If G is a vector valued measure defined on Σ_0 which is absolutely continuous with respect to m, that is $\lim_{m(E)\to 0} G(E) = 0$, then it can be extended to a measure on Σ, being still absolutely continuous with respect to m. (See [**D**],[**DU**])

We refer the reader to ([**DU**], [**D**]) and to ([**J**], [**GC-RF**]) for general theory and the properties we shall use about vector valued measures and Hardy spaces respectively.

2. Vector valued measures of bounded mean oscillation

Definition 2.1. Let $1 \leq q < \infty$. Given a countably additive measure G defined on Σ and with values in X, it is said that G belongs to $MBMO_q(\Omega, X)$ if

$$|G|_{*,q} = \sup\{(\sum_{i=1}^{n} \|\frac{G(E_i)}{m(E_i)} - \frac{G(B)}{m(B)}\|^q \frac{m(E_i)}{m(B)})^{1/q}\} < \infty \tag{2.1}$$

where the supremun is taken over all balls B and over all finite partitions of B in pairwise disjoint measurable sets E_i with $m(E_i) > 0$.

When $m(\Omega) = \infty$ then $\|G\|_{MBMO_q} = |G|_{*,q}$ gives a norm on the set of equivalence classes of measures: $G_1 \sim G_2$ if there is b in X such that $G_1(E) - G_2(E) = b\,m(E)$ for all measurable set E .

For $m(\Omega) < \infty$ we consider the norm $\|G\|_{MBMO_q} = |G|_{*,q} + \|G(\Omega)\|$.

It is obvious that if $1 < q_1 < q_2 < \infty$ then

$$V^{\infty}(\Omega, X) \subset MBMO_{q_2}(\Omega, X) \subset MBMO_{q_1}(\Omega, X) \tag{2.2}$$

Remark 2.1. Let us assume G belong to $MBMO_q(\Omega, X)$. Given a ball B and a measurable set $E \subset B$, it is quite immediate to find a constant C_B depending on B satisfying

$$\|G(E)\| \leq C_B \max(m(E), m(E)^{1-1/q}) \tag{2.3}$$

Suposse we consider $B_n = \{y \in \Omega : d(x_0, y) < n\}$ and denote by G_{B_n} the measure G concentrated on B_n, that is $G_{B_n}(E) = G(E \cap B_n)$. A glance at (2.3) allows us to say that for any $1 < q < \infty$ if G belongs to $MBMO_q(\Omega, X)$ then G_{B_n} are necessarily absolutely continuous with respect to m and this clearly implies that also G is absolutely continuous with respect to m. (Recall that for vector-measures on σ-algebras it suffices to check that they vanish on m-null sets).

Proposition 2.1. *Let $1 \leq q < \infty$, g be locally in $L_q(\Omega, X)$ and G be an X-valued measure such that $G(E) = \int_E g(x)\, dm(x)$ for all measurable bounded set E.*

Then g belongs to $BMO_q(\Omega, X)$ if and only if G belongs to $MBMO_q(\Omega, X)$. Moreover $\|G\|_{MBMO_q} = \|g\|_{BMO_q}$.

Proof: Given any ball B, consider $G_B(E) = G(E \cap B) - \frac{G(B)}{m(B)} m(E \cap B)$. Observe that

$$\sup\{(\sum_{i=1}^{n} \|\frac{G(E_i)}{m(E_i)} - \frac{G(B)}{m(B)}\|^q \frac{m(E_i)}{m(B)})^{1/q} : (E_i) \text{ partition of } B\}$$

coincides with the q-variation of G_B on Ω divided by $m(B)^{1/q}$ and G_B is a measure represented by the function $(g - g_B)\chi_B$, that is

$$G_B(E) = \int_E (g(x) - g_B)\chi_B \, dm(x).$$

Therefore the proposition follows from the equality between the q-variation and the norm in L_q of the function which represents the measure (see [**D**]). ∎

Remark 2.2. In general it is not true that any measure in $MBMO_q(\Omega, X)$ is representable by a function, this depends on the Radon-Nikodym property. We refer the reader to [**B1**] for the case $\Omega = \{|z| = 1\}$, but a similar result and proof can be established also in this general setting.

Proposition 2.2. *Let $1 \leq q < \infty$. G belongs to $MBMO_q(\Omega, X)$ if and only if there exists a family of vectors in X, say $\{a_B$: B ball$\}$, such that*

$$\sup\{(\sum_{i=1}^{n} \|\frac{G(Ei)}{m(E_i)} - a_B\|^q \frac{m(Ei)}{m(B)})^{1/q}\} < \infty \tag{2.4}$$

where the supremum is taken over all balls B and over all finite partitions of B in pairwise disjoint measurable sets E_i with $m(E_i) > 0$

Proof: The direct implication is obvious by taking $a_B = \frac{G(B)}{m(B)}$. To show the converse let us assume that we have $\{a_B$: B ball$\}$ with the above property, and notice that

$$\|a_B - \frac{G(B)}{m(B)}\| \leq C$$

for all B (simply take the partition of B given only by B).

Therefore for any B and any partition

$$(\sum_{i=1}^{n} \|\frac{G(E_i)}{m(E_i)} - \frac{G(B)}{m(B)}\|^q \frac{m(E_i)}{m(B)})^{1/q} \leq$$

$$(\sum_{i=1}^{n} \|\frac{G(E_i)}{m(E_i)} - a_B\|^q \frac{m(E_i)}{m(B)})^{1/q} + (\sum_{i=1}^{n} \|a_B - \frac{G(B)}{m(B)}\|^q \frac{m(E_i)}{m(B)})^{1/q} \leq C \quad ∎$$

As in the case of functions we can define an equivalent norm in $MBMO_q(\Omega, X)$. Let us take

$$|G|'_{*,q} = \sup_{ball B} \{\inf_{a \in X} \frac{1}{m(B)^{1/q}} |G - a\,m|_q(B)\}. \tag{2.5}$$

Note that essentially the same argument as in Proposition 2.2. shows the following

$$|G|'_{*,q} \leq |G|_{*,q} \leq C\,|G|'_{*,q} \tag{2.6}$$

Proposition 2.3. *Let $1 < q < \infty$. If G belongs to $MBMO_q(\Omega, X)$ then there exists a non negative function ϕ in $BMO_q(\Omega)$ such that*

$$|G|(E) = \int_E \phi(x)\, dm(x).$$

Moreover $\|\phi\|_{BMO_q} \leq C\, \|G\|_{MBMO_q}$.

Proof: Since G is countably additive and m-continuous then the same is true for the variation of G, $|G|$. Therefore using the Radon-Nikodym theorem there exists a non negative measurable function ϕ which represents the measure $|G|$. To show that ϕ belongs to $BMO_q(\Omega)$, we shall use Propositions 2.2 and 2.1. We simply have to find a family of real numbers $\{a_B\}$ such that

$$\sup\{(\sum_{i=1}^{n} | \frac{|G|(Ei)}{m(Ei)} - a_B |^q \frac{m(E_i)}{m(B)})^{1/q}\} < \infty$$

Take $a_B = \frac{\|G(B)\|}{m(B)}$, and observe that

$$\left| |G|(E) - \frac{\|G(B)\|}{m(B)} m(E) \right| \leq |G - \frac{G(B)}{m(B)} m|(E)$$

Then

$$\sup\{(\sum_{i=1}^{n} | \frac{|G|(Ei)}{m(Ei)} - \frac{\|G(B)\|}{m(B)} |^q \frac{m(E_i)}{m(B)})^{1/q}\} \leq$$

$$\sup\{\frac{1}{m(B)^{1/q}} (\sum_{i=1}^{n} (|G - \frac{G(B)}{m(B)} m|(E_i))^q m(E_i)^{1-q})^{1/q}\} \leq |G|_{*,q}$$

The last inequality follows from Remark 1.1. ■

3. The theorem and its proof

In the sequel $1 < p, q < \infty$, with $\frac{1}{p} + \frac{1}{q} = 1$. In this section we shall achieve the duality result between $H^1_p(\Omega, X)$ and $MBMO_q(\Omega, X^*)$. We shall need several lemmas before we prove the result. The next result was done in [**B1**] for the circle and for $q = 2$, and here we present a different approach which is valid for general spaces of homogeneous type. The author would like to point out that a similar and independent proof of the following lemma has been obtained by T. Wolniewicz (personal communication).

Lemma 3.1. *Let G be a measure in $MBMO_q(\Omega, X)$. Then for each integer $n \in \mathbf{N}$ we can find a measure G_n in $V^\infty(\Omega, X)$ and a constant C_n satisfying $|G_n|_{*,q} \leq C_n$ and such that*

$$|G|_{*,q} \leq \lim_{n\to\infty} C_n \leq K|G|_{*,q} \tag{3.1}$$

$$\lim_{n\to\infty} G_n(E) = G(E) \qquad \textit{for all measurable bounded set } E. \tag{3.2}$$

Proof: Using Proposition 2.3 we first get a function ϕ in $BMO_q(\Omega)$.

Denote by $\Omega_n = \{x \in X : \phi(x) > n\}$ and $\phi_n(x) = \min(1, n/\phi(x))$. Let us define now

$$G_n(E) = \int_E \phi_n(x)\, dG(x) \qquad (E \in \Sigma_0) \tag{3.3}$$

Notice that

$$\|G_n(E)\| \leq |G_n|(E) \leq \int_E \phi_n(x)\, d|G|(x) \leq \int_E \phi_n(x)\, \phi(x)\, dm(x) \leq n\, m(E)$$

This, using Remark 1.2., allows to extend G_n to Σ and shows that G_n belongs to $V^\infty(\Omega, X)$.

On the other hand

$$G(E) - G_n(E) = \int_{E\cap\Omega_n} (1 - \phi_n(x)) dG(x) \tag{3.4}$$

Therefore if E is contained in some ball B

$$\|G(E) - G_n(E)\| \leq 2 \int_{E\cap\Omega_n} \phi(x) dm(x)$$

Since $\phi\chi_B$ is in $L_1(\Omega)$ then taking limit as $n \to \infty$ shows (3.2).

From (2.6) we have finally to estimate $m(B)^{-1/q}|G_n - a\,m|_q(B)$ for all balls B. Using (3.4) we have that for any $E \subset B$

$$\|G(E) - G_n(E)\| \leq \int_{E\cap\Omega_n} (1 - n/\phi(x))\, d|G|(x)$$

If $\|a\| \leq n$ then

$$\|G(E) - G_n(E)\| \leq \int_{E\cap\Omega_n} (\phi(x) - n)\, dm(x) \leq \int_{E\cap\Omega_n} (\phi(x) - \|a\|)\, dm(x)$$

Therefore we have

$$|G_n - G|_q(B) \leq |G - a\,m|_q(B \cap \Omega_n) \tag{3.5}$$

Though $|G|_q$ is not a measure for $q > 1$ the q-variation es subadditive and therefore we get that for all $\|a\| \leq n$

$$m(B)^{-1/q}|G_n - a\,m|_q(B) \leq 2\,m(B)^{-1/q}|G - a\,m|_q(B) \tag{3.6}$$

Denoting now by

$$D_n = \sup_{ball B} \inf_{\|a\|\leq n} \{m(B)^{-1/q}|G - a\,m|_q(B)\}$$

we get (3.1) for $C_n = 2\,C\,D_n$ where C is the constant appearing in (2.6). ■

Notice that $V^\infty(\Omega, X^*)$ can be obviously identified with the dual of $L_1(\Omega, X)$. Indeed any measure G in $V^\infty(\Omega, X^*)$ defines a functional T_G acting on X-valued simple functions (which are dense in $L_1(\Omega, X)$) by the formula

$$T_G(\sum_{i=1}^n a_i \chi_{E_i}) = \sum_{i=1}^n < G(E_i), a_i > \tag{3.7}$$

where $<,>$ means duality between X and X^*.

Lemma 3.2. *Let $1 < p, q < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$ and G belong to $V^\infty(\Omega, X^*)$. Then*

$$|T_G(f)| \leq C\|G\|_{MBMO_q} \|f\|_{H_p^1} \text{ for all } f \text{ in } H_p^1(\Omega, X). \tag{3.8}$$

Proof: Let us first take a "simple atom" in $H_p^1(\Omega, X)$, that is
$s = \sum_{i=1}^n b_i \chi_{E_i}$, $E_i \subset B$ for some ball B, $\sum_{i=1}^n b_i\, m(E_i) = 0$
and $\sum_{i=1}^n \|b_i\|_X^p\, m(E_i) \leq m(B)^{1-p}$.
For such an atom we can write

$$T_G(\sum_{i=1}^n b_i \chi_{E_i}) = \sum_{i=1}^n < G(E_i), b_i >= \sum_{i=1}^n < G(E_i) - \frac{G(B)}{m(B)} m(E_i), b_i >$$

Therefore

$$|T_G(s)| \leq \sum_{i=1}^n \|G(E_i) - \frac{G(B)}{m(B)} m(E_i)\|_{X^*} m(E_i)^{-1/p} m(E_i)^{1/p} \|b_i\|_X \leq$$

$$\leq (\sum_{i=1}^n \|\frac{G(E_i)}{m(E_i)} - \frac{G(B)}{m(B)}\|_{X^*}^q m(E_i))^{1/q} (\sum_{i=1}^n \|b_i\|_X^p\, m(E_i))^{1/p} \leq$$

$$\leq (\sum_{i=1}^n \|\frac{G(E_i)}{m(E_i)} - \frac{G(B)}{m(B)}\|_{X^*}^q \frac{m(E_i)}{m(B)})^{1/q} \leq |G|_{*,q}$$

For a general atom a supported in B in $H_p^1(\Omega, X)$ we can use approximation by simple functions in $L_p(\Omega, X)$, and find a sequence of simple functions d_k supported in B converging to a in $L_p(\Omega, X)$, and take the sequence $s_k = \big(d_k - \int_B d_k(x)\, dm(x)\big)\chi_B$ which clearly also converges to a in $L_p(\Omega, X)$. Hence $\|s_k\|_p \leq 2\|a\|_p$ for k large enough, and therefore $s_k/2$ are "simple atoms".

Using now that T_G is continuous as operator on $L_1(\Omega, X)$, and that s_k converges to a in $L_1(\Omega, X)$, then

$$|T_G(a)| = \lim_{k\to\infty} |T(s_k)| = 2 \lim_{k\to\infty} |T(s_k/2)| \leq 2\,|G|_{*,q} \tag{3.9}$$

For a general function f, take any representation of f in $H_p^1(\Omega, X)$, say $f = \sum_{j=0}^{\infty} \lambda_j a_j$, where the a_j are (X,p)-atom and $\sum_{j=0}^{\infty} |\lambda_j| < \infty$ and notice that (3.8) follows from (3.9) and the fact that the series $f = \sum_{j=0}^{\infty} \lambda_j a_j$ is absolutely convergent in $L_1(\Omega, X)$ what implies that $T_G(f) = \sum_{j=0}^{\infty} \lambda_j\, T_G(a_j)$. ■

Theorem 3.1. *Let* $1 < p, q < \infty$ *and* $\frac{1}{p} + \frac{1}{q} = 1$. *Then*

$$(H_p^1(\Omega, X))^* = MBMO_q(\Omega, X^*) \; (\textit{equivalent norms}) \tag{3.10}$$

Proof: Let us take G in $MBMO_q(\Omega, X^*)$, and define as above

$$T_G\Big(\sum_{i=1}^{n} b_i \chi_{E_i}\Big) = \sum_{i=1}^{n} < G(E_i), b_i >$$

From the definition of $H_p^1(\Omega, X)$ we can easily see that simple functions with support in balls are dense in the space, therefore it is enough to see that

$$|T_G\Big(\sum_{i=1}^{n} b_i \chi_{E_i}\Big)| \leq C\,|G|_{*,q}\, \|\Big(\sum_{i=1}^{n} b_i \chi_{E_i}\Big)\|_{H_p^1} \tag{3.11}$$

To see (3.11) we first invoke Lemma 3.1 to find a sequence of measures G_n in $V^\infty(\Omega, X^*)$, that according to (3.2) verifies $\lim_{n\to\infty} T_{G_n}(s) = T_G(s)$ for all simple function supported in a ball.

Secondly we use Lemma 3.2, together with (3.1) to get

$$|T_G(s)| \leq \lim_{n\to\infty} |T_{G_n}(s)| \leq C \lim_{n\to\infty} |G_n|_{*,q} \|s\|_{H_p^1} \leq$$

$$C \lim_{n\to\infty} C_n \,\|s\|_{H_p^1} \leq C\,|G|_{*,q} \|s\|_{H_p^1}.$$

For the converse we shall deal first with the case $m(\Omega) < \infty$. Let us take now a functional T in $\big(H_p^1(\Omega, X)\big)^*$. Since constant functions are also considered as X-atoms in the case of finite measure we have that $a\chi_E \in H^1_p(\Omega, X)$, what allows us to define the following X^* valued measure.

$$(3.12) \qquad < G(E), a >= T(a\chi_E) \qquad (\text{a} \in \text{X})$$

Given a ball B and a partition of B, say $\{E_i\}$, of pairwise disjoint sets, using the duality $(l^p(X))^* = l^q(X^*)$, we have

$$(\sum_{i=1}^n \|\frac{G(E_i)}{m(E_i)} - \frac{G(B)}{m(B)}\|^q_{X^*} \frac{m(E_i)}{m(B)})^{1/q} =$$

$$(\sum_{i=1}^n \|(\frac{G(E_i)}{m(E_i)} - \frac{G(B)}{m(B)})(\frac{m(E_i)}{m(B)})^{1/q}\|^q_{X^*})^{1/q} =$$

$$\sup\{\sum_{i=1}^n < (\frac{G(E_i)}{m(E_i)} - \frac{G(B)}{m(B)})(\frac{m(E_i)}{m(B)})^{1/q}, b_i > | : \sum_{i=1}^n \|b_i\|^p_X = 1\}.$$

On the other hand we have

$$|\sum_{i=1}^n < (\frac{G(E_i)}{m(E_i)} - \frac{G(B)}{m(B)})(\frac{m(E_i)}{m(B)})^{1/q}, b_i > | =$$

$$\frac{1}{m(B)^{1/q}} |\sum_{i=1}^n < \frac{G(E_i)}{m(E_i)^{1/p}}, b_i > - < \frac{G(B)}{m(B)}, \sum_{i=1}^n m(E_i)^{1/q} b_i) > | =$$

$$= \frac{1}{m(B)^{1/q}} |T(\sum_{i=1}^n m(E_i)^{-1/p} b_i \chi_{E_i}) - T(b\chi_B)|$$

where $b = \frac{1}{m(B)}(\sum_{i=1}^n m(E_i)^{1/q} b_i)$.

Denote by $a = \frac{1}{2\, m(B)^{1/q}}(\sum_{i=1}^n m(E_i)^{-1/p} b_i \chi_{E_i} - b\chi_B)$. It is elementary to show that if $\sum_{i=1}^n \|b_i\|^p_X = 1$ then a is a (X,p)-atom.

Therefore we obtain

$$(\sum_{i=1}^n \|\frac{G(E_i)}{m(E_i)} - \frac{G(B)}{m(B)}\|^q_{X^*} \frac{m(E_i)}{m(B)})^{1/q} \le 2\,|T(a)| \le 2\,\|T\|$$

This shows $|G|_{*,q} \le 2\,\|T\|$. Since T and T_G coincide over simple atoms, we have $T = T_G$.

On the other hand

$$\|G(\Omega)\| \le \sup\{ \quad |T(b\chi_\Omega)| : \|b\| \le 1 \quad \} \le m(\Omega)\,\|T\|$$

and this finishes the proof for the finite measure case.

Let us deal now with the case of $m(\Omega) = \infty$. Take a functional T in $\left(H^1_p(\Omega, X)\right)^*$ and a ball B in Ω. Let us consider the following space

$$L^p_0(B,X) = \{f \in L_p(\Omega, X) : \text{ supp } f \subset B \text{ and } \int_B f(x)\,dm(x) = 0\}$$

The following function is an (X,p)-atom

$$a(x) = \frac{f(x)}{m(B)^{1/q}\|f\|_p} \; for \; f \in L^p_0(B,X).$$

hence

$$\|f\|_{H^1_p} \leq m(B)^{1/q}\,\|f\|_p$$

and therefore

$$\|Tf\| \leq \|T\|\,m(B)^{1/q}\,\|f\|_p$$

This shows that T defines a bounded functional on $L^p_0(B,X)$ and hence from the Hahn-Banach extension theorem, we get an element in the dual of $L_p(B,X)$. The characterization of the dual space $\left(L_p(B,X)\right)^*$ in terms of X^*-valued measures of bounded q-variation allows us to find a measure G_B with values in X^* verifying

$$T(f) = \int_B f\,dG_B \qquad f \in L^p_0(B,X) \tag{3.13}$$

(Note that this measure is uniquely determined up to a measure $F(E) = \xi\, m(E \cap B)$ for some $\xi \in X^*$). Now if we take an increasing sequence of balls converging to Ω, say B_n, and we determine G_{B_n} by the assumption $G_{B_n}(B_1) = 0$, then we can construct a vector-valued measure on Σ_0, given by $G(E) = G_{B_n}(E)$ for $E \subset B_n$. It is clear that G_{B_n} are absolutely continuous and hence the same is true for G. Now from remark 1.2 we get an extension to Σ.

$$\left(\sum_{i=1}^{n} \left\|\frac{G(E_i)}{m(E_i)} - \frac{G(B)}{m(B)}\right\|^q_{X^*} \frac{m(E_i)}{m(B)}\right)^{1/q} = \sup_{\|f\|_p=1} \left|\frac{1}{m(B)^{1/q}} \int_B f\, d(G - \frac{G(B)}{m(B)} m)\right|$$

For each $f \in L_p(B,X)$, consider $a = \frac{1}{2\,m(B)^{1/q}}(f - f_B)\chi_B$ and therefore

$$\left(\sum_{i=1}^{n} \left\|\frac{G(E_i)}{m(E_i)} - \frac{G(B)}{m(B)}\right\|^q_{X^*} \frac{m(E_i)}{m(B)}\right)^{1/q} = \sup_a 2\,|T(a)| \leq 2\,\|T\|$$

This completes the proof. ■

Remark 3.1. For $1 < p, r < \infty$,

$$MBMO_q(\Omega, X) = MBMO_r(\Omega, X) \text{ with equivalent norms}$$

For dual spaces follows from the theorem and (1.1), and the general case is consequence of the embedding $X \subset X^{**}$

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