TRABAJOS DE INVESTIGACION OPERATIVA
Vol. 7, núm. 1, 1990, pp. 51 a 61

CONTROL OF LINEAR SYSTEMS WITH RATIONAL EXPECTATIONS. THE CASE OF INCOMPLETE INFORMATION

EMILIO CERDA TENA
Depto. de Análisis Económico
Universidad Complutense de Madrid

SUMMARY

The problem of optimal control of linear economic systems with rational expectations and quadratic objective function is solved for the case of incomplete information. The case of complete information has been previously studied. In both problems the hypothesis of causality is not satisfied and, therefore, the standard techniques of control theory cannot be directly applied, though the method used is based on these techniques.
Key words: Incomplete information, Rational expectations, Stochastic control.
AMS Classification: 93A99.

1. INTRODUCTION

In most non-trivial economic decisions the time variable intervenes. Many economic agents' decisions at a particular moment in time depend on their view of the future, that is to say, on their expectations. The same applies with the behaviour of the economic aggregates making up macroeconomic models. In the economic literature there exist different ways of modelling expectations, the most important being: static expectations, adaptive and rational ones, with the latter having acquired great importance in recent years, and now being a necessary reference in any treatise on Dynamic Economics (Pesaran (1987), Aoki (1989), Holly and Hughes Hallet (1989)).

Recibido: Febrero 1990.
Revisado: Mayo 1991.

The rational expectations hypothesis, in Muth's strong version, assumes that the expectations that the economic agents has at moment t, on the value that a variable will take in the future (which is subjective and non-observable), is the mathematical expectation of the variable conditioned by the information which is held in t, implied by the model. The hypothesis assumes, therefore, that individuals act as if they knew the model and formed their expectations in accordance with it.

The control theory techniques are not applicable, in general, to systems with rational expectations of the state variables, since the causality hypothesis is not met. (Aoki-Canzoneri (1979), Chow (1980), Driffill (1981), Buiter (1983)).

In Cerdá (1990) a method is presented, based on Dynamic Programming, which solves the control problem of a linear systems with rational expectations and a quadratic objective function. In thi article we are going to deal with the problem for the case of incomplete information: it is assumed that the state variables (endogenous variables) are not observable, but there exist observation variables related to them by the observation equation.

In section 2 the problem is set out. In section 3 some preliminary results are obtained. In section 4 the theorem for solving the problem is presented. In section 5 the article's conclusions are presented.

2. STATEMENT OF THE PROBLEM

We consider the problem with the following objective function, state equation and observation equation:

$$\min E_0 W = E_0 \sum_{t=1}^{T} (y_t - a_t)' K_t (y_t - a_t), \quad K_t \text{ being a positive semidefinite, symmetric matrix.}$$

$$y_t = B_t y^*_{t/t-1} + B_{1t} y^*_{t+1/t-1} + A_t y_{t-1} + C_t x_t + b_t + u_t \quad (\text{for } t = 1, \dots, T) \tag{1}$$

$$z_t = M_t y_t + w_t \quad (\text{for } t = 0, \dots, T)$$

where:

y_t is a non-observable vector of endogenous variables.

x_t is a vector of control variables.

b_t is a vector which records the combined effects of exogenous variables not subject to control.

z_t is a vector of observable variables.

We assume that $y_0, u_1, ..., u_T, w_0, ..., w_T$ are random mutually uncorrelated vectors, so that:

$$Eu_t = 0 \quad ; \quad Eu_t u_t' = U_t$$
$$Ew_t = 0 \quad ; \quad Ew_t w_t' = W_t$$
$$Ey_0 = m \quad ; \quad E(y_0 - m)(y_0 - m)' = S$$

Moreover, we assume that W_t is positive definite for each t.

$y^*_{t/k} = E(y_t|I_k)$: is the (rational) expectation that is held at the end of the period k on the value that the vector y will take in period t. In this case $I_k = \{z_k, ..., z_0; x_k, ..., x_1; b_k, ..., b_1\}$ but does not contain $y_k, ..., y_0$, since they are unknown (non-observable).

$$E_0 W = E(W|I_0)$$

We assume that the exogenous variables $\{b_t\}$ are stochastic, in the form:

$$b_t = \sum_{i=1}^{p} R_i b_{t-i} + \xi_t$$

where $\{\xi_t\}$ is a serially uncorrelated stochastic process, of zero mean, which is independent of the disturbances that enter the system explaining y_t, of the observation noises and of the random variable y_0 (p can be considered finite or infinite).

Note: In its most general form the system (1) would be,

$$y_t = B_t y^*_{t/t-1} + \sum_{i=1}^{p} B_{it} y^*_{t+i/t-1} + A_t y_{t-1} + C_t x_t + b_t + u_t \quad (\text{for} \quad t = 1, \quad ..., \quad T)$$

In this article the case $p = 1$ is developed, since if $p \neq 1$ the treatment is analogous.

3. PRELIMINARY RESULTS

3.1. Expression of the system in another equivalent form

Proposition 1

Let us consider system (1). Let us assume that, for each t, it is verified that the matrix $(I - B_t)$ is non singular. Thus, this system can be explained in the following way

$$y_t = \tilde{B}_{1t} y^*_{t+1/t-1} + A_t J_{t-1} + (\tilde{A}_t - A_t) E(y_{t-1}|I_{t-1}) + \tilde{C}_t x^*_{t/t-1} + \tilde{b}^*_{t/t-1} + v_t$$

where:

$$\tilde{B}_{1t} = (I - B_t)^{-1} B_{1t}$$
$$\tilde{A}_t = (I - B_t)^{-1} A_t$$
$$\tilde{C}_t = (I - B_t)^{-1} C_t$$
$$\tilde{b}^*_{t/t-1} = (I - B_t)^{-1} b^*_{t/t-1}$$
$$v_t = C_t(x_t - x^*_{t/t-1}) + (b_t - b^*_{t/t-1}) + u_t$$

Proof:

We consider system (1)

$$y_t = B_t y^*_{t/t-1} + B_{1t} y^*_{t+1/t-1} + A_t y_{t-1} + C_t x_t + b_t + u_t \quad (\text{for } t = 1, 2, ..., T)$$

where, as we have pointed out in section 2, expectations are rational, so that $y^*_{t/t-1} = E[y_t|I_{t-1}]$; $y^*_{t+1/t-1} = E[y_{t+1}|I_{t-1}]$.

Taking the expectations conditioned to I_{t-1} on the two sides of system (1) and bearing in mind that the expectations are rational, we obtain:

$$y^*_{t/t-1} = B_t y^*_{t/t-1} + B_{1t} y^*_{t+1/t-1} + A_t E(y_{t-1}|I_{t-1}) + C_t x^*_{t/t-1} + b^*_{t/t-1} \Rightarrow$$
$$\Rightarrow y^*_{t/t-1} = (I - B_t)^{-1} [B_{1t} y^*_{t+1/t-1} + A_t E(y_{t-1}|I_{t-1}) + C_t x^*_{t/t-1} + b^*_{t/t-1}]$$

therefore:

$$y_t = y^*_{t/t-1} + A_t y_{t-1} - A_t E(y_{t-1}|I_{t-1}) + v_t =$$
$$= \tilde{B}_{1t} y^*_{t+1/t-1} + A_t y_{t-1} + (\tilde{A}_t - A_t) E(y_{t-1}|I_{t-1}) + \tilde{C}_t x^*_{t/t-1} + \tilde{b}^*_{t/t-1} + v_t$$

Corollary

In the case of complete information, we have

$$I_k = \{y_k, ..., y_0; x_k, ..., x_1; b_k, ..., b_1\}$$

so that

$$y_t = \tilde{B}_{1t} y^*_{t+1/t-1} + \tilde{A}_t y_{t-1} + \tilde{C}_t x^*_{t/t-1} + \tilde{b}^*_{t/t-1} + v_t$$

a result that coincides with the one we had in that case (Cerdá, 1990).

Note: We assume, as in the case of complete information, that the vectors v_t are uncorrelated in time and have zero mean. Moreover, y_0, v_1, ..., v_T, w_0, ..., w_T are mutually uncorrelated. Hence $Ev_t = 0$; we call $Ev_r v'_t = V_t$.

3.2. Previous problem

Before setting out and proving the theorem that solves the problem we are dealing with, we are going to raise another problem of control with incomplete information, for a particular formulation of the system, without expectations, the solution of which we will use in an auxiliary way in proving the theorem which most concerns us.

The «previous problem» is the following one:

$$\min E_0 W = E_0 \sum_{t=1}^{T} (y_t - a_t)' K_t (y_t - a_t),$$ K_t being a positive semidefinite, symmetric matrix.

$$y_t = D_t y_{t-1} + (\tilde{A}_t - D_t) E(y_{t-1} | I_{t-1}) + \tilde{C}_t x_t + \tilde{b}_t + u_t \quad \text{for } t = 1, ..., T$$

$$z_t = M_t y_t + w_t \quad \text{for } t = 0, ..., T$$

We assume that y_0, u_1, ..., u_T, w_0, ..., w_T are uncorrelated. Moreover, $Eu_t = 0$; $Ev_t = 0\ \forall t$

$$I_k = \{z_k, ..., z_0; x_k, ..., x_1; \tilde{b}_k, ..., \tilde{b}_1\}$$

The exogenous variables $\{\tilde{b}_t\}$ are of the form:

$$\tilde{b}_t = \sum_{i=1}^{q} R_i \tilde{b}_{t-i} + \eta_t$$

where $\{\eta_t\}$ is a stochastic process, of zero mean, serially uncorrelated, which is independent of the disturbances that enter the system that explains y_t, of the observation noises and of the random variable y_0.

3.3. Solution of the previous problem

Theorem 1

The solution to the problem set out in 3.2 is the following:

$$\hat{x}_t = G_t E(y_{t-1}|I_{t-1}) + g_t$$

where $\begin{cases} G_t = -(\tilde{C}_t' H_t \tilde{C}_t)^{-1} \tilde{C}_t' H_t \tilde{A}_t \\ g_t = -(\tilde{C}_t' H_t \tilde{C}_t)^{-1} \tilde{C}_t' (H_t \tilde{b}^*_{t/t-1} - h^*_{t/t-1}) \end{cases}$

being $\begin{cases} \mathrm{H}_{t-1} = \mathrm{K}_{t-1} + (\tilde{\mathrm{A}}_t + \tilde{\mathrm{C}}_t \mathrm{G}_t)' \mathrm{H}_t (\tilde{\mathrm{A}}_t + \tilde{\mathrm{C}}_t \mathrm{G}_t), & \text{with } H_T = K_T \\ h_{t-1} = K_{t-1} a_{t-1} + (\tilde{A}_t + \tilde{C}_t G_t)' (h^*_{t/t-1} - H_t \tilde{b}^*_{t/t-1}), & \text{with } h_T = K_T a_T \end{cases}$

The proof of this theorem appears in Cerdá (1987).

3.4. Commentaries on the solution obtained

Let us consider the problem of standard linear-quadratic control, with complete information (problem SIC):

$$\min E_0 W = E_0 \sum_{t=1}^{T} (y_t - a_t)' K_t (y_t - a_t),$$ K_t being a positive semidefinite, symmetric matrix.

$$y_t = \tilde{A}_t y_{t-1} + \tilde{C}_t x_t + \tilde{b}_t + u_t, \quad \text{for } t = 1, \ldots, T$$

where the variables b_t are stochastic and of the form previously expressed.

The optimal solution of this problem is $\hat{x}_t = G_t y_{t-1} + g_t$ where G_t and g_t and the corresponding H_t and h_t are given by the very same expressions in theorem 1.

The incomplete information version of the same problem (problem SII)

$$\min E_0 W = E_0 \sum_{t=1}^{T} (y_t - a_t)' K_t (y_t - a_t),$$ K_t being a positive semidefinite, symmetric matrix.

$$y_t = \tilde{A}_t y_{t-1} + \tilde{C}_t x_t + \tilde{b}_t + u_t, \quad \text{for } t = 1, ..., T$$
$$z_t = M_t y_t + w_t \quad \text{for } t = 0, ..., T$$

has as a solution $\hat{x}_t = G_t E(y_{t-1}|I_{t-1}) + g_t$, G_t, g_t being analagous to those obtained in theorem 1, the same as H_t, h_t and, therefore, also analagous to those of problem SIC. Consequently, this problem and the so-called previous problem have exactly the same solution. Moreover, in the expression of the optimal control of the previous problem, and thus, of problem SII, we find exactly the expression of the optimal control of problem SIC, with the sole change of $E(y_{t-1}|I_{t-1})$ in place of y_{t-1}.

The complete developments of the optimal solutions of the problems SIC and SII, with notations and assumptions used in this article appear in Cerdá (1987).

4. SOLUTION TO THE PROBLEM

The following theorem gives us the solution to the problem we are dealing with.

Theorem 2

Let us consider the problem set out in section 2. We assume that, for $t = T$ (final period), it is verified that $y^*_{T+1/T-1} = \Gamma y^*_{T/T-1}$.

The optimal solution is $\hat{x}_t = F_t E[y_{t-1}|I_{t-1}] + f_t$, where F_t, f_t coincide with the expressions calculated in the case of complete information.

Proof

Our starting point is the version in complete information of the problem we are dealing with and its optimal solution (Cerdá, 1990). For this case, the system can be expressed:

$$y_t = \tilde{B}_{1t} y^*_{t+1/t-1} + \tilde{A}_t y_{t-1} + \tilde{C}_t x^*_{t/t-1} + \tilde{b}^*_{t/t-1} + v_t$$

where $\tilde{B}_{1t}$. $\tilde{A}_t$, $\tilde{C}_t$, $\tilde{b}^*_{t/t-1}$, v_t coincide with the expressions appearing in proposition 1.

In point 2 of the method used to solve the problem in the case of complete information, $y^*_{t+1/t-1}$ is treated as given and using dynamic programming, we obtain:

$$\hat{x}^*_{t/t-1} = G_t y_{t-1} + G_{1t} y^*_{t+1/t-1} + g_t$$

In the case we are dealing with now (incomplete information), we have seen in proposition 1 that the system can be expressed as:

$$y_t = \tilde{B}_{1t} y^*_{t+1/t-1} + A_t y_{t-1} + (\tilde{A}_t - A_t) E(y_{t-1}|I_{t-1}) + \tilde{C}_t x^*_{t/t-1} + \tilde{b}^*_{t/t-1} + v_t \quad (2)$$

We now treat $y^*_{t+1/t-1}$ as given and use theorem 1 along with the results commented on in 3.4, obtaining:

$$\hat{x}^*_{t/t-1} = G_t E(y_{t-1}|I_{t-1}) + G_{1t} y^*_{t+1/t-1} + g_t \quad (3)$$

where, G_t, G_{1t}, g_t coincide with the expressions obtained in the case of complete information.

Taking this result to (2), we obtain:

$$y_t = R_{1t} y^*_{t+1/t-1} + A_t y_{t-1} + (R_t - A_t) E(y_{t-1}|I_{t-1}) + r_t + v_t \quad (4)$$

where

$$R_{1t} = \tilde{B}_{1t} + \tilde{C}_t G_{1t} \quad ; \quad R_t = \tilde{A}_t + \tilde{C}_t G_t \quad ; \quad r_t = \tilde{b}^*_{t/t-1} + \tilde{C}_t g_t$$

(it should be observed that $r_t = r^*_{t/t-1}$, but in general, $r_t \neq r^*_{t/t-j}$, for $j > 1$).

We are going to solve this system. We will prove by induction that system (4) can be expressed as:

$$y_t = A_t y_{t-1} + (P_t - A_t) E(y_{t-1}|I_{t-1}) + s_t + v_t \quad (5)$$

where $P_t = (I - R_{1t} P_{t+1})^{-1} R_t$ for $t = 1, ..., T$ with $P_{T+1} = \Gamma$

$$s_t = (I - R_{1t} P_{t+1})^{-1} (r_t + R_{1t} s^*_{t+1/t-1})$$

for $t = 1, ..., T$, with $s_{T+1} = 0$.

- For T

By particularising (4) for $t = T$, and bearing in mind that $y^*_{T+1/T-1} = \Gamma y^*_{T/T-1}$, we get:

$$y_T = R_{1T} \Gamma y^*_{T/T-1} + A_T y_{T-1} + (R_T - A_T) E(y_{T-1}|I_{T-1}) + r_T + v_T$$

Taking expectations conditioned to I_{T-1} on the two sides of this equation and bearing in mind that expectations are rational, we obtain:

$$y^*_{T/T-1} = R_{1T}\Gamma y^*_{T/T-1} + R_T E(y_{T-1}|I_{T-1}) + r_T$$

Then:

$$\begin{aligned} y_T &= y^*_{T/T-1} + A_T y_{T-1} - A_T E(y_{T-1}|I_{T-1}) + v_T = \\ &= A_T y_{T-1} + (P_T - A_T)E(y_{T-1}|I_{T-1}) + s_T + v_{T'} \end{aligned}$$

where:

$$\begin{aligned} P_T &= (I - R_{1T}\Gamma)^{-1} R_T \\ s_T &= (I - R_{1T}\Gamma)^{-1} r_T \end{aligned}$$

- Let us assume that (5) is true for $t + 1$. Then

$$y_{t+1} = A_{t+1} y_t + (P_{t+1} - A_{t+1})E(y_t|I_t) + s_{t+1} + v_{t+1}$$

We are going to prove it for t.

We start from equation (4).

From the starting point of the induction hypothesis for $t + 1$, and taking expectations conditioned to I_{t-1} we get:

$$y^*_{t+1/t-1} = P_{t+1} y^*_{t/t-1} + s^*_{t+1/t-1}$$

Taking this result to (4) and, taking expectations conditioned to I_{t-1} we obtain:

$$\begin{aligned} y^*_{t/t-1} = {} & R_{1t} P_{t+1} y^*_{t/t-1} + R_{1t} s^*_{t+1/t-1} + A_t E(y_{t-1}|I_{t-1}) + \\ & + (R_t - A_t)E(y_{t-1}|I_{t-1}) + r_t \end{aligned}$$

Then:

$$\begin{aligned} y_t &= y^*_{t/t-1} + A_t y_{t-1} - A_t E(y_{t-1}|I_{t-1}) + v_t = \\ &= A_t y_{t-1} + (P_t - A_t)E(y_{t-1}|I_{t-1}) + s_t + v_t \end{aligned}$$

where

$$\begin{aligned} P_t &= (I - R_{1t} P_{t+1})^{-1} R_t \\ s_t &= (I - R_{1t} P_{t+1})^{-1} (r_t + R_{1t} s^*_{t+1/t-1}) \end{aligned}$$

This is the solution of the system we were looking for.

The values of P_t, s_t that we have obtained coincide with those calculated in the case of complete information.

We have, therefore,

$$y_t = A_t y_{t-1} + (P_t - A_t)E(y_{t-1}|I_{t-1}) + s_t + v_t \tag{6}$$
$$\Rightarrow y_{t+1} = A_{t+1}y_t + (P_{t+1} - A_{t+1})E(y_t|I_t) + s_{t+1} + v_{t+1}$$
$$\Rightarrow y^*_{t+1/t-1} = P_{t+1}y^*_{t/t-1} + s^*_{t+1/t-1}$$

From (6) we obtain

$$y^*_{t/t-1} = P_t E(y_{t-1}|I_{t-1}) + s_t$$

Therefore,

$$y^*_{t+1/t-1} = P_{t+1}P_t E(y_{t-1}|I_{t-1}) + P_{t+1}s_t + s^*_{t+1/t-1}$$

Taking this result to (3) we have

$$\hat{x}^*_{t/t-1} = (G_t + G_{1t}P_{t+1}P_t)E(y_{t-1}|I_{t-1}) + g_t + G_{1t}P_{t+1}s_t + G_{1t}s^*_{t+1/t-1}$$

Therefore,

$$\hat{x}^*_{t/t-1} = \hat{x}_t = F_t E(y_{t-1}|I_{t-1}) + f_t$$

where

$$F_t = G_t + G_{1t}P_{t+1}P_t$$
$$f_t = g_t + G_{1t}(P_{t+1}s_t + s^*_{t+1/t-1})$$

We thus see that F_t, f_t coincide with the expressions obtained in the case of complete information and the theorem is proved.

Note: In the proof of the theorem we have assumed that the exogenous variables $\{b_t\}$ are stochastic and of the type

$$b_t = \sum_{i=1}^{p} R_i b_{t-i} + \xi_t$$

The theorem is equally true for the case of deterministic variables $\{b_t\}$ and the proof is analogous.

5. CONCLUSIONS

In the above pages the problem of control of linear systems with quadratic objective function and finite time horizon, for the case of incomplete information, has been set out and solved. In Cerdá (1990) the corresponding case of complete information is solved.

Although the usual Control Theory techniques are not applicable to his case since they do not meet the causality hypothesis, the final result which is obtained preserves the property known in standard linear-quadratic problems: the expression of the optimal control for the case of incomplete information exactly coincides with that obtained in complete information, with the sole change being y_{t-1} for $E(y_{t-1}/I_{t-1})$.

In the Gaussian case this conditional expectation $E(y_{t-1}/I_{t-1})$ will be calculated by using the Kalman's filter for the state equation and the observation system.

REFERENCES

AOKI, M. (1989): *Optimization of Stochastic Systems. Topics in Discrete-Time Dynamics*, Second Edition, Academic Press.

AOKI, M., and CANZONERI, M. (1979): «Reduced forms of rational expectations models», *The Quarterly Journal of Economics*, February, pp. 59-71.

BUITER, W. (1983): «Expectations and control theory», *Economie appliquée*, n.° 1, pp. 129-156.

CERDA, E. (1987): *Control estocástico de modelos con expectativas racionales*, Tesis Doctoral, Universidad Complutense.

CERDA, E. (1990): «Control óptimo de sistemas lineales con expectativas racionales», *Investigaciones económicas* (Segunda época), vol. XIV, n.° 1, pp. 63-83.

CHOW, G. (1980): «Econometric policy evaluation and optimization under rational expectations», *Journal of Economic Dynamics and Control*, vol. 2, n.° 1, pp. 47-59.

DRIFFILL, E. J. (1981): «Time inconsistency and "rules vs. discretion" in macroeconomic models with rational expectations», *Discussion Papers in Economics and Econometrics*, University of Southampton.

HOLLY, S., and HUGHES HALLET, A. (1989): *Optimal control, expectations and uncertainty*, Cambridge University Press.

PESARAN, M. H. (1987): *The limits to rational expectations*, Basil Blackwell.