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LIMITED SPACE DOUBLE CHANNEL MARKOVIAN QUEUE WITH HETEROGENEOUS SERVERS

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ABSTRACT

For a double channel markovian queue with finite waiting space and unequal service rates at the two counters, the difference equations satisfied by the Laplace transforms of the state probabilities at finite time are solved and the state probabilities have been obtained. The closed form of the state probabilities can be used to obtain the important parameters of the system.

Key words: Markovian Queue. Heterogeneous Servers, Double Channel.

1. INTRODUCTION

Recently Sharma and Gupta [1] have obtained a closed form solution for the transient behaviour of an $M/M/1/N$ queue. In this paper we discuss $M/M/2/N$ queueing system with unidentical services rates at the two channels. There are many real life problems conforming to such a model for example a heterogeneous multiprocessor system or a medical clinic with two service counters of unequal serving capacity. The closed form solution is obtained and results for $M/M/1/N$ model can be derived as a particular case by putting $\mu_2 = 0$. Also by putting $\mu = \mu_2$ we get the results for $M/M/2/N$ queue having equal service rates at both the counters.

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2. THE MODEL

We consider a $M/M/2/N$ queueing system with inter-arrival time being negative exponentially distributed with parameterλ and service time distributions also being negative exponential with parameters μ_1 and μ_2 respectively. Without loss of generality assume that $\mu_1 > \mu_2$ which also implies that first arriving unit in the queue (when it is empty) joins the first counter for service and thereafter the arriving unit goes to the counter which it finds free. Again the waiting room capacity is taken limited to $N - 2$ places, i.e., the maximum number of customers in the system is restricted to N. Furthermore we assume that there are i customers waiting at the time $t = 0$ when the service starts and the traffic intensity $\rho = \lambda/(\mu_1 + \mu_2)$.

Let $p_n(t)$ be the probability that there are n customers in the system at the time t. Then $p_i(0) = 1$ and $p_n(0) = 0 \, \forall \, n \neq i$. Writing the difference-differential equations of the system and taking Laplace transform of these equations, we get

$$
\begin{aligned}
(\lambda + \theta)\psi(0,\theta) &= \mu_1\psi(1,\theta) + \delta_{io} \\
(\lambda + \mu_1 + \theta)\psi(1,\theta) &= \lambda\psi(0,\theta) + (\mu_1 + \mu_2)\psi(2,\theta) + \delta_{il} \qquad (2.1) \\
(\lambda + \mu_1 + \mu_2 + \theta)\psi(n,\theta) &= \lambda\psi(n-1,\theta) + (\mu_1 + \mu_2)\psi(n+1,\theta) + \delta_{in} \\
& \qquad n = 2, 3, \ldots, N-1 \\
(\mu_1 + \mu_2 + \theta)\psi(N,\theta) &= \lambda\psi(N-1,\theta) + \delta_{iN}
\end{aligned}
$$

where δ_{ij} is the usual kronecker delta and

$$\psi(n,\theta) = \int_0^\infty e^{-\theta t} p^n(t)\, dt$$

Equations (2.1) can be written as

$$A\Psi = [\delta_{i0} \quad \delta_{i1} \quad \delta_{i2} \quad \cdots \quad \delta_{iN}]' \qquad (2.2)$$

where A is $N + 1 \times N + 1$ matrix given by

$$A = \begin{bmatrix} \lambda+\theta & -\mu_1 & 0 & 0 & \cdots & 0 & 0 \\ -\lambda & \lambda+\theta+\mu_1 & -(\mu_1+\mu_2) & 0 & \cdots & 0 & 0 \\ 0 & -\lambda & \lambda+\theta+\mu_1\mu_2 & -(\mu_1+\mu_2) & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & -\lambda & \theta+\mu_1+\mu_2 \end{bmatrix} \qquad (2.3)$$

and $\Psi = [\psi(0,\theta)\psi(1,\theta)\psi(2,\theta)\cdots\psi(N,\theta)]'$.

Now using Crammer's rule we get

$$\psi(n,\theta) = \frac{D_n(\theta)}{D(\theta)},\ n = 0, 1, 2, ..., N. \tag{2.4}$$

where $D(\theta) = |A|$ and $D_n(\theta)$ is obtained from $D(\theta)$ by replacing its n^{th} column by the column on the right hand side of (2.2). We now evaluate $D_n(\theta)$ and $D(\theta)$. By setting $\lambda + \theta + \mu_1 + \mu_2 = x$ and $\lambda(\mu_1 + \mu_2) = c$ and after little use of properties of determinants we obtain

$$D(\theta) = \theta f_N(x) \tag{2.5}$$

where

$$f_N(x) = \begin{vmatrix} x-\mu_2 & \sqrt{c-\lambda\mu_2} & 0 & 0 & 0 & \cdots & 0 & 0 \\ \sqrt{c-\lambda\mu_2} & x & \sqrt{c} & 0 & 0 & \cdots & 0 & 0 \\ 0 & \sqrt{c} & x & \sqrt{c} & 0 & \cdots & 0 & 0 \\ 0 & 0 & \sqrt{c} & x & \sqrt{c} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & \sqrt{c} & x \end{vmatrix} \tag{2.6}$$

with $f_0(x) = 1$.
Now $f_N(x)$ is an N^{th} degree polynomial in x and its zeros are the eingevalues of the symmetric tridiagonal matrix B given by

$$\sqrt{c}B = \sqrt{c}\begin{bmatrix} -\mu_2/\sqrt{c} & \sqrt{(c-\lambda\mu_2)}/\sqrt{c} & 0 & 0 & \cdots & 0 & 0 \\ \sqrt{(c-\lambda\mu_2)}/\sqrt{c} & 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 & 0 \end{bmatrix} \tag{2.7}$$

These eigenvalues are real and distinct [2] and let us denote them by $\alpha_{N,k}$ $(k = 1, 2, ..., N)$, so that we can write

$$f_N(x) = \prod_{k=1}^{N} (x + \alpha_{Nk}\sqrt{c})$$

and let

$$F_N(\theta) = \prod_{k=1}^{N} (\theta + \lambda + \mu_1 + \mu_2 + \alpha_{Nk}\sqrt{\lambda(\mu_1 + \mu_2)}) \tag{2.8}$$

Now in order to evaluate $D_n(\theta)$ we set

$$g_n(x) = \begin{vmatrix} x & \sqrt{c} & 0 & 0 & \cdots & 0 & 0 \\ \sqrt{c} & x & \sqrt{c} & 0 & \cdots & 0 & 0 \\ 0 & \sqrt{c} & x & \sqrt{c} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & \sqrt{c} & x \end{vmatrix}_{n \times n} \tag{2.9}$$

and

$$h_n(x) = \begin{vmatrix} x-\mu_1-\mu_2 & \sqrt{c-\lambda\mu_2} & 0 & 0 & \cdots & 0 & 0 \\ \sqrt{c-\lambda\mu_2} & x-\mu_2 & \sqrt{c} & 0 & \cdots & 0 & 0 \\ 0 & \sqrt{c} & x & \sqrt{c} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & \sqrt{c} & x \end{vmatrix}_{n \times n} \tag{2.10}$$

with $g_o(x) = h_o(x) = 1$
and denote $\beta_{n,k}$, $\gamma_{n,k}$ as the zeros of the polynomials $g_n(x)$ and $h_n(x)$ and again after some algebra we get

$$\begin{aligned} D_n(\theta) &= (1-\delta_{io})[\mu_1(\mu_1+\mu_2)^{i-1}\{G_{N-i}(\theta) - \lambda G_{N-i-1}(\theta)\}] + \\ &\quad + \delta_{io}[G_N(\theta) - (\mu_2+\lambda)G_{N-1}(\theta) + \lambda\mu_2 G_{N-2}(\theta)], \quad n = 0 \\ &= (\mu_1+\mu_2)^{i-n}[G_{N-i}(\theta) - \lambda G_{N-i-1}(\theta)]H_n(\theta) \quad n = 1, 2, \dots, i \\ &= \lambda^{n-i}[G_{N-n}(\theta) - \lambda G_{N-n-1}(\theta)]H_i(\theta), \quad n = i+1, i+2, \dots, N \end{aligned} \tag{2.11}$$

where

$$G_n(\theta) = \prod_{k=1}^{n} (\theta + \lambda + \mu_1 + \mu_2 + \beta_{n,k}\sqrt{\lambda(\mu_1+\mu_2)})$$

$$\mathrm{H}_n(\theta) = \prod_{k=1}^{n} (\theta + \lambda + \mu_1 + \mu_2 + \gamma_{n,k}\sqrt{\lambda(\mu_1+\mu_2)})$$

As $D(\theta)$ has real distinct factors, we make use of partial fractions and taking inverse Laplace transform of (2.4), we get after considerable simplification for $\lambda \neq \mu_1 + \mu_2$

$$p_n(t)\begin{cases} = \mu_1 B + e^{-(\lambda_1+\mu_1+\mu_2)t}\displaystyle\sum_{j=1}^{N}[(1-\delta_{io})\{(-1)^i\frac{\mu_1}{\lambda}\rho^{\frac{i}{2}+1}A_{ij}e^{-\alpha_{Nj}t\sqrt{c}}\}+\delta_{io}A_{oj}e^{-\alpha_{Nj}t\sqrt{c}}], & n=0 \\ = (\mu_1+\mu_2)B\rho^n + e^{-(\lambda+\mu_1+\mu_2)t}\displaystyle\sum_{j=1}^{N}(-1)^{n-i}\rho^{(n-i)/2}A_{ij}h_n(\alpha_{Nj})e^{-\alpha_{Nj}t\sqrt{c}}, & n=1,2,\dots,i \\ = (\mu_1+\mu_2)B\rho^n + e^{-(\lambda+\mu_1+\mu_2)t}\displaystyle\sum_{j=1}^{N}(-1)^{n-i}\rho^{(n-i)/2}A_{nj}h_i(\alpha_{Nj})e^{-\alpha_{Nj}t\sqrt{c}}, & n=i+1,i+2,\dots,N \end{cases} \tag{2.12}$$

where

$$B = (1-\rho)\{\mu_1(1-\rho)+\lambda(1-\rho^N)\}^{-1}$$

$$A_{nj} = \frac{g_{N-n}(\alpha_{Nj}) + \sqrt{\rho}\,g_{N-n-1}(\alpha_{Nj})}{(\rho^{1/2}+\rho^{-1/2}+\alpha_{Nj})b_{Nj}}$$

$$A_{oj} = \frac{g_N(\alpha_{Nj}) + (\rho^{1/2}+\mu_2/\sqrt{c})g_{N-1}(\alpha_{Nj}) + \mu_2/(\mu_1+\mu_2)g_{N-2}(\alpha_{Nj})}{(\rho^{1/2}+\rho^{-1/2}+\alpha_{Nj})b_{Nj}}$$

$$b_{Nj} = \prod_{\substack{k=1\\k\neq j}}^{N}(\alpha_{Nj}-\alpha_{Nk})$$

and for $\lambda = \mu_1+\mu_2$ ($\rho = 1$)

$$p_n(t)\begin{cases} = \dfrac{\mu_1}{\mu_1+\lambda N} + e^{-2\lambda t}\displaystyle\sum_{j=1}^{N}[(1-\delta_{io})\{(-1)^i\frac{\mu_1}{\lambda}A'_{ij}e^{-\alpha_{Nj}\lambda t}\}+\delta_{i0}A'_{0j}e^{-\alpha_{Nj}\lambda t}], & n=0 \\ = \dfrac{\mu_1}{\mu_1+\lambda N} + e^{-2\lambda t}\displaystyle\sum_{j=1}^{N}(-1)^{n-i}A'_{ij}h_n(\alpha_{Nj})e^{-\alpha_{Nj}\lambda t}, & n=1,2,\dots,i \\ = \dfrac{\mu_1}{\mu_1+\lambda N} + e^{-2\lambda t}\displaystyle\sum_{j=1}^{N}(-1)^{n-i}A'_{nj}h_i(\alpha_{Nj})e^{-\alpha_{Nj}\lambda t}, & n=i+1,i+2,\dots,N \end{cases} \tag{2.13}$$

where

$$A'_{oj} = \frac{g_N(\alpha_{Nj}) + (1+\mu_2/\lambda)g_{N-1}(\alpha_{Nj}) + \mu_2/\lambda\, g_{N-2}(\alpha_{Nj})}{(2+\alpha_{Nj})b_{Nj}}$$

$$A'_{nj} = \frac{g_{N-n}(\alpha_{Nj}) + g_{N-n-1}(\alpha_{Nj})}{(2+\alpha_{Nj})b_{Nj}}$$

By letting $t \to \infty$, we get the steady state distribution from (2.12) and (2.13) and it is given by

$$p_n = \begin{cases} \dfrac{\mu_1(1-\rho)}{\mu_1(1-\rho)+\lambda(1-\rho^N)} , & n=0 \\ \dfrac{(\mu_1+\mu_2)(1-\rho)\rho^n}{\mu_1(1-\rho)+\lambda(1-\rho^N)} , & n=1,2,...,N \end{cases} , \rho \neq 1$$

$$p_n = \begin{cases} \dfrac{\mu_1}{\mu_1 + \lambda N} , & n=0 \\ \dfrac{\lambda}{\mu_1 + \lambda N} , & n=1,2,...,N \end{cases} , \rho = 1 \qquad (2.14)$$

Furthermore, when $t \to \infty$, $N \to \infty$ and $\rho < 1$, we get the well known steady state distribution given by

$$p_n = \begin{cases} \dfrac{\mu_1(1-\rho)}{\mu_1(1-\rho)+\lambda} , & n = 0 \\ \dfrac{(\mu_1+\mu_2)(1-\rho)\rho^n}{\mu_1(1-\rho)+\lambda} , & n \parallel 1 \end{cases} \qquad (2.15)$$

Using $p_n(t)$ from (2.12) and (2.13) the parameters of the system such as queue length, variance, number of customers waiting in the system at a time t can be easily worked out. Again putting $\mu_1 = \mu_2 = \mu$ in the above results we obtain results for $M/M/2/N$ queue with identical service at both the counters and by putting $\mu_2 = 0$ we can derive results for $M/M/1/N$ queue which tally with the results reported in the literature [1].

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