TRABAJOS DE INVESTIGACION OPERATIVA
Vol. 4. Núm. 1, 1989 pp. 3 a 10

# MULTIPARAMETRIC LINEAR FRACTIONAL FUNCTIONALS PROGRAMMING

CHADA, S. S.
Universidad de Wisconsin Eau Claire
Eau-Claire, Wisconsin 54701, USA

## RESUMEN

En este trabajo se genera un programa de funciones multiparamétricas líneo-fraccionales cuyos parámetros aparecen sólo en la función objetiva. La solución óptima de este programa paramétrico trata de mantener los límites únicamente como ecuaciones. Se muestra asimismo que la serie de parámetros forman un poliedro convexo.

**Palabras clave:** optimización, programa paramétrico, programa lineo-fraccional.
**Clasificación AMS:** 90C31 (programación paramétrica).
**Título:** Funciones de programación paramétrica líneo-fraccional.

## ABSTRACT

In this paper a multiparametric linear fractional functionals program, with parameters appearing only in the objective function, is generated. The optimum solution of this parametric program is supposed to satisfy the constraints as equations only. It is also shown that the set of parameters forms a convex polyhedron.

**Key words:** optimization, parametric programming, linear fractional programming.
**AMS Classification:** 90C31 (Parametric Programming).

Recibido: julio 1986.
Aceptado: noviembre 1988.

## 1. INTRODUCCION

The importance of multiparametric linear programming is well known and this problem has been studied by many including Gal, Neduma (1972). Aggarwal (1968) and Chadha (1983, 1971) have obtained some interesting results for a class of parametric linear fractional functionals programming problems. Present note examines the possibility of maintaining the optimum solution of a linear fractional functionals programming problem while not allowing one or more of the constraints to be satisfied as an inequality or inequalities. We delete the constraints, which hold as inequalities at an optimum solution, and combine them with the objective function to obtain a multiparametric linear fractional functionals programming problem. We investigate the behavior of the optimum solution with regards to this multiparametric objective function. It is also shown that the set of parameters, $[\lambda_1, \lambda_2, ..., \lambda_r, \mu_1, \mu_2, ..., \mu_r]$, which maintains the optimality of the solution forms a convex polyhedron. Constraints which are satisfied as inequalities are named as invalid constraints. A simple numerical example is also worked out.

## 2. A MULTIPARAMETRIC PROBLEM

Consider the following linear fractional functionals programming problem:

$$\text{Maximize} = \frac{CX}{DX + \alpha}$$

$$\text{Subject to } X \in S.$$

Here:

$S = [X : \bar{A}X \leqslant\ =\ \geqslant b,\ X \geqslant 0;$
$C = (c_1, c_2, ..., c_n)$ and
$D = (d_1, d_2, ..., d_n)$ are row vectors in $R^n$;
$\alpha$ is a positive scalar,
$b = (b_1, b_2, ..., b_m)$ is a column vector in $R^m$
$\bar{A} = (a_{ij})$ is an $m$ by $n$ matrix, and
$X = (x_1, x_2, ..., x_n)$ is a column vector of the variables in $R^n$.

After introducing slack and surplus variables, the above problem will look like:

$$\text{Maximize } Z = \frac{CX}{DX + \alpha}$$
$$\text{Subject to } AX = b \qquad (1)$$
$$X \geqslant 0 \, .$$

Without any loss of generality, we assume that first $m$ columns of $A$ constitute the optimal basis, $B$, for problem (1). Furthermore, let the rows of $A$ be so arranged that the last $r$ rows represent invalid constraints. Under the usual assumptions that the set $S$ is regular and that $DX + \alpha > 0$ over $S$; a basic feasible solution $X_B = B^{-1}b$ is optimal if Martos (1964), Swarup (1965),

$$\Delta_j = Z_2[Z_j^1 - c_j] - Z_1[Z_j^2 - d_j] \geqslant 0$$

for all js.

Here

$$Z_2 = D_B X_B + \alpha, \qquad Z_1 = C_B X_B$$
$$Z_j^1 = C_B Y_j, \qquad Z_j^2 = D_B Y_j$$

and

$$Y_j = B^{-1} a_j, \; a_j \text{ is the } j^{\text{th}} \text{ column of } A.$$

We delete invalid constraints and combine them with the objective function and obtain the following multiparametric linear fractionals programming problem.

$$\text{Maximize } Z = \frac{\sum_{i=1}^{n} c_i x_i \pm \lambda_1 \sum_{j=1}^{n} a_{m-r+1\,j} x_j + \cdots \pm \lambda_r \sum_{j=1}^{n} a_{mj} x_j}{\sum_{i=1}^{n} d_i x_i \pm \mu_1 \sum_{j=1}^{n} a_{m-r+1\,j} x_j + \cdots \pm \mu_r \sum_{j=1}^{n} a_{mj} x_j + \alpha}$$

Subject to

$$\sum_{j=1}^{n} a_{ij} x_j = b_i$$
$$x_j \geqslant 0; \qquad (2)$$
$$i = 1, 2, \ldots, (m - r).$$

$\lambda$'s and $\mu$'s are parameters. Positive and negative coefficients correspond to less than and greater than constraints respectively. For multiparametric problem (2), we prove the following.

**Theorem:** The basic feasible solution $X_B = B^{-1}b$, disregarding all slack and surplus variables at positive level, will maintain the optimality of the problem if

$$\overset{*}{\Delta}_j = \Delta_j - Z_2 \sum_{i=i}^{r} \lambda_i y_{m-r+ij} + Z_1 \sum_{i=i}^{r} \mu_i y_{m-r+ij} \geqslant 0$$

$$\text{for } j = m + 1, ..., n.$$

Furthermore, the set of parameters $(\lambda_1, \lambda_2, ..., \lambda_r, \ \mu_1, ..., \mu_r)$, which maintains the optimality of the solution, forms a convex polyhedron.

**Proof:** We prove the result for «<» invalid constraints (proof for «>» invalid constraints follows similarly). For a basic feasible solution to be optimal we must have

$$\overset{*}{\Delta}_j = \overset{*}{Z}_2[\overset{*}{Z}{}_j^1 - \overset{*}{c}_j] - \overset{*}{Z}_1[\overset{*}{Z}{}_j^2 - \overset{*}{d}_j] \geqslant 0 \tag{3}$$

$$j = m + 1, ..., n.$$

Here

$$\overset{*}{Z}_2 = (d_1 + \mu_1 a_{m-r+ll} + \cdots + \mu_r a_{ml})x_1 + \cdots + \cdots +$$

$$+ (d_{m-r} + \mu_1 a_{m-r+lm-r} + \cdots + \mu_r a_{mm-r})x_{m-r} + \alpha =$$

$$= \sum_{i=1}^{m-r} d_i x_i + \alpha + \mu_i \sum_{i=1}^{m-r} a_{m-r+li} x_i + \cdots + \mu_r \sum_{i=1}^{m-r} a_{mi} x_i$$

$d_i'$s associated with the slack and surplus variables are zeros and the matrix $(a_{ij})i = m - r + 1, ..., m; j = 1, 2, ..., m - r$ is a null matrix therefore:

$$\overset{*}{Z}_2 = \sum_{i=1}^{m} d_i x_i + \alpha = Z_2 \tag{4}$$

Also

$$\overset{*}{Z}_1 = (c_1 + \lambda_1 a_{m-rll} + \cdots + \lambda_r a_{ml})x_1 + \cdots + \cdots +$$

$$+ (c_{m-r} + \lambda_1 a_{m-r+lm-r} + \cdots + \lambda_r a_{mm-r})x_{m-r} =$$

$$= \sum_{i=1}^{m-r} c_i x_i + \lambda_l \sum_{i=1}^{m-r} a_{m-r+li} x_i + \cdots + \lambda_r \sum_{i=1}^{m-r} a_{mi} x_i =$$

$$= \sum_{i=1}^{m} c_i x_i + 0 = Z_1 \qquad (5)$$

Let us consider

$$\overset{*}{Z}{}_j^1 - c_j^* = (c_1 + \lambda_1 a_{m-r11} + \cdots + \lambda_r a_{ml})y_{1j} + \cdots +$$

$$+ \cdots + (c_{m-r} + \lambda_{1m-r+lm-r} + \cdots + \lambda_r a_{mm-r})y_{m-rj} -$$

$$- (c_j + \lambda_1 a_{m-r+lj} + \cdots + \lambda_r a_{mj}) =$$

$$= \sum_{i=1}^{m-r} c_i y_{ij} - c_j + \lambda_1 \left( \sum_{i=1}^{m-r} a_{m-r+li} y_{ij} - a_{m-r+lj} \right) + \cdots +$$

$$+ \lambda_r \left( \sum_{i=1}^{m-r} a_{mi} y_{ij} - a_{mj} \right)$$

Now

i) $a_{sj} = y_1 j a_{s1} + \cdots + y_{mj} a_{sm}$
for $j = m + 1, \ldots, n$ and
$s = 1, 2, \ldots, m.$

ii) $c_i's$ associated with slack and surplus variables are zeros and

iii) the matrix $(a_{ij})$, $i = m - r + 1, \ldots, m$; $j = m - r + 1, \ldots, m$
is an $r$ by $r$ identity matrix;

therefore,

$$\overset{*}{Z}{}_j^1 - \overset{*}{c}_j = \left( Z_j^1 - c_j - \sum_{i=m-r+1}^{m} c_i y_{ij} + \cdots + \right.$$

$$+ \lambda_1 \left( a_{m-r+lj} - \sum_{i=m-r+1}^{m} a_{m-r+li} y_{ij} - a_{m-r+lj} \right) +$$

$$+ \cdots + \lambda_r\left(a_{mj} - \sum_{i=m-r+1}^{m} a_{mi}y_{ij} - a_{mj}\right) =$$

$$= Z_j^1 - c_j - \sum_{i=1}^{r} \lambda_i y_{m-r+ij} \tag{6}$$

Similar arguments yield that

$$\overset{*}{Z}{}_j^2 - d_j = Z_j^2 - d_j + \sum_{i=1}^{r} \mu_i y_{m-r+ij} \tag{7}$$

$$j = m + 1, \ldots, n.$$

Using results (4) to (7) in (3) we get

$$\overset{*}{\Delta}_j = Z_2\left[Z_j^1 - c_j - \sum_{i=1}^{r} \lambda_i y_{m-r+ij}\right] - Z_1\left[Z_j^2 - d_j - \sum_{i=1}^{r} \mu_i y_{m-r+ij}\right] =$$

$$= Z_2[Z_j^1 - c_j] - Z_1[Z_j^2 - d_j] - Z_2 \sum_{i=1}^{r} \lambda_i y_{m-r+ij} +$$

$$+ Z_1 \sum_{i=1}^{r} \mu_i y_{m-r+ij} = \Delta_j - Z_2 \sum_{i=1}^{r} \lambda_i y_{m-r+ij} + Z_1 \sum_{i=1}^{r} \mu_i y_{m-r+ij}$$

$$j = m + 1, \ldots, n.$$

This should be $\geqslant 0$ for the solution to remain optimal.

Each of $\overset{*}{\Delta}_j$ is a linear non-homogeneous inequality in the 2r parameters $(\lambda_1, \lambda_2, \ldots, \lambda_r, \mu_1, \ldots, \mu_r)$ and thus represents a half space in $E^{2r}$. The $\lambda$'s and $\mu$'s which simultaneously satisfy all the conditions must, therefore. form a convex polyhedron.

## 3. NUMERICAL EXAMPLE

$$\text{Maximize } Z = \frac{x_1 + 3x_2 + 2x_3}{2x_1 + x_2 + 4x_3 + 1}$$

Subject to

$$x_1 + 3x_2 + 6x_3 \leqslant 8$$
$$2x_1 + x_2 + 4x_3 \leqslant 5$$
$$x, x_2, x_3 \geqslant 0$$

The final table for $x_2 = 8/3$, $x_1 = x_3 = 0$ to be an optimal solution is

| | $a_1$ | $a_2$ | $a_3$ | $a_4$ | $a_5$ | $X_B$ |
|---|---|---|---|---|---|---|
| | 1/3 | 1 | 2 | 1/3 | 0 | 8/3 |
| | 5/3 | 0 | 2 | 1/3 | 1 | 7/3 |
| $\Delta_j \rightarrow$ | 40/3 | | 92/3 | 1 | | |

For the solution $x_2 = \frac{8}{3}$, $x_1 = x_3 = 0$, we observe that first constraint

$$\frac{1}{3}x_1 + x_2 + 2x_3 \leqslant \frac{8}{3}$$

holds as an equality, while second constraint

$$\frac{5}{3}x_1 + 0x_2 + 2x_3 \leqslant \frac{7}{3}$$

holds as an inequality. Thus second constraint is an invalid constraint. We combine this invalid constraint with the objective function and generate the following parametric programming problem.

$$\text{Maximize } Z = \frac{x_1 + 3x_2 + 2x_3 + \lambda(5/3x_1 + 2x_3)}{2x_1 + x_2 + 4x_3 + 1 + \mu(5/3x_1 + 2x_3)}$$

Subject to

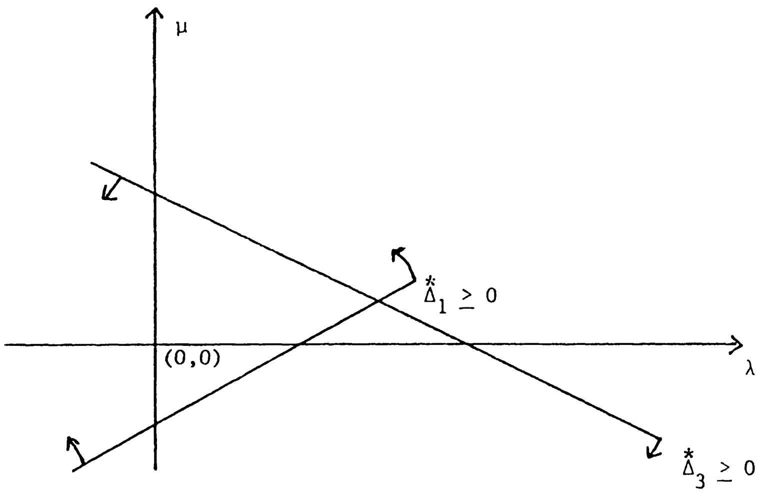
$$1/3x_1 + x_2 + 2x_3 = 8/3$$
$$x_1, x_2, x_3 \geqslant 0$$

For the solution $x_2 = 8/3$, $x_1 = x_3 = 0$ to remain optimal, we must have

$$\overset{*}{\Delta}_1 = 40/3 - 55/9\lambda + 40/3\mu \geqslant 0$$

$$\overset{*}{\Delta}_3 = 92/3 - 22/3\lambda + 16\mu \geqslant 0.$$

Suitable selection of $(\lambda, \mu)$ can be made in accordance with the above criteria (shaded region shown below) to keep the solution optimal while not allowing any invalid constraints.




## REFERENCES

MARTOS, B. (1964): «Hyperbolic Programming. (Translated by Andrew and Verconika Whinston.)», *Naval Research Log. Quart. II*, 135-155.

SWARUP, K. (1965): «Linear Fractional Functionals Programming», *Opera tions Research, 13*, 1029-1036.

AGGARWAL, S. P. (1968): «Parametric Linear Fractional Functionals Programming», *Metrika*, 12, 106-114.

CHADHA, S. A. (1983): «Quadratic Parametric Linear Programming», *Cahiers du C.E.R.O., 25*, 23-28.

CHADHA, S. A. (1971): «A Linear Fractional Functionals Program with a Two Parameter Objective Function», *ZAMM, 51*, 479-482.

GAL, T., and NEDOMA, J. (1972): «Multiparametric Linear Programming», *Management Science*, 18.