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DUALITY THEOREMS FOR A CLASS OF NON-LINEAR PROGRAMMING PROBLEMS

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RESUMEN

La dualidad de la programación lineal se usa para establecer un importante teorema de dualidad para una clase de problemas de programación no-lineal. El problema primario tiene una función objetiva cuasimonotónica y un polihedro convexo como su limitación (constraint set).

Palabras clave: dualidad, programación no-lineal, función cuasimonotónica.

Clasificación AMS: 90C30 (programación no-lineal).

Title: Duality Theorems for a Class of Non-Linear Programming Problems

SUMMARY

Duality of linear programming is used to establish an important duality theorem for a class of non-linear programming problems. Primal problem has quasimonotonic objective function and a convex polyhedron as its constraint set.

Key words: duality, non-linear programming, quasimonotonic function.

AMS classification: 90C30 (non-linear programming).

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1. INTRODUCTION

The concept of duality is investigated briefly for a class of nonlinear programming problems. We are interested in a nonlinear programming problem whose quasimonotonic objective function needs to be optimized over a constraint set formed by linear inequalities. Results proved here generalize the duality results of Kaska (1969). The following minimization problem is taken as the primal problem (P-P).

$$\begin{aligned} &\text{Minimize } f(x) \\ &\text{Subject to } x \in S \\ &\text{Where } S = (x : Ax \geqslant b, x \geqslant 0) \end{aligned} \tag{1}$$

The nonlinear function $f(x)$ is quasimonotonic over the set S; $A = (a\ a_2, ..., a_m)$ is an $m \cdot n$ matrix; x and b are column vectors with n and m components respectively.

Before we formulate a dual of problem (1), we state and prove a lemma.

Lemma 1

A basic feasible solution $x_0 = (x'_B, 0)'$ is an optimal solution to problem (1) if

$$f_{x_j}(x_0) - Y'_j f_{x_B}(x_0) \geqslant 0 \tag{2}$$

for all colums a_j of the matrix (A, I) in S. $f_x(x)$ in the $n \cdot 1$ gradient vector of $f(x)$ at x' over a matrix is used to denote its transpose. $Y_j = B^{-1} a_j$, where B is the basis matrix corresponding to the basic feasible solution x_0.

Proof:

$f(x)$ is quasimonotonic over the set S. It follows from Martos (1965) that $f(x)$ will attain its minimum at an extreme point of the set S.

Let $x = (x'_B, 0)'$ be a basic feasible solution of problem (1).

In the case when x does not minimize $f(x)$, then an improved value of $f(x)$ (assuming non-degeneracy) can be obtained by inserting some column a_j into the basis and by deleting column a_k from the basis. The new basic feasible solution, $\hat{x}$ is computed from Hadley (1963)

$$\hat{x} = x_B - \theta Y_j$$

$$x_j = \frac{x_{Bk}}{y_{kj}} = \theta$$

where

$$Y_j = B^{-1} a_j = (y_1, y_2, ..., y_m)'.$$

Inserting the 0-components corresponding to the non-basic variables, above can be written as

$$\hat{x} = x - \theta \bar{Y}_j + \theta e_j$$

here

$$\bar{Y}_j = (Y_j', 0)'$$

$f(x)$ is quasimonotonic, therefore from Martos (1965) it follows that $f(\hat{x}) \geqslant f(x)$ implies that

$$\begin{aligned}(\hat{x} - x)' f_x(x) &= \theta(e_j - \bar{Y}_j)' f_x(x) \\ &= \theta[f_{x_j}(\hat{x}) - Y_j' f_{x_B}(\hat{x})] \geqslant 0.\end{aligned}$$

Under the non-degeneracy assumptions, if for some j,

$$f_{x_j}(\hat{x}) - Y_j' f_{x_B}(\hat{x}) < 0$$

then $f(x) > f(\hat{x})$. This shows that the insertion of the a_j-th column into the basis decreases the value of the objective function. Thus if $x_0' = (x_B', 0)'$ is a minimizing solution of the problem (1), then

$$f_{x_j}(x_0) - Y_j' f_{x_B}(x_0) \geqslant 0$$

should hold true for all $j's = 1, 2, ..., n$. This proves the lemma.

The dual problem, (D-P), is now defined as

$$\begin{aligned} &\text{Maximize } F(u,v) = f(u) \\ &\text{Subject to } A'v - f_x(u) \leqslant 0 \qquad (3) \\ &\qquad -b'v + u'[f_x(u)] \leqslant 0 \qquad (4) \\ &\qquad u, v \geqslant 0 \end{aligned}$$

Let constraint set of (D-P) be denoted by T.

2. THE DUALITY THEOREMS

Theorem 1:

Let G be the infimum of $f(x)$ over S, and g be the supremum of $F(u,v)$ over T, then

$$g \leqslant G$$

Proff:

We adopt the convention that $G = +\infty$ if the set T is null and that $g = -\infty$ if the set S is null. It is, therefore, sufficient to prove theorem for the case when T and S are non-null.

Let $x \in S$ and $(u,v) \in T$. From (3) it follows that

$$v'Ax \leqslant [f_x(u)]'x \qquad (5)$$

Also from $Ax \geqslant b$, we have, $v'Ax \geqslant v'b$. This fact along with (4) when used in (5) yields

$$(x-u)'f_x(u) \geqslant 0 \qquad (6)$$

$f(x)$ is quasimonotonic, therefore, from (6) it follows that

$$f(x) \geqslant f(u) = F(u,v)$$

Theorem 2:

Let x_0 be a feasible solution for (P-P), and (u_0, v_0) be a feasible solution for (D-P), such that

$$f(x_0) = F(u_0, v_0).$$

Then x_0 will minimize $f(x)$ and (u_0, v_0) will maximize $F(u, v)$.

Proff:

From theorem 1, for any $x \in S$, we have

$$f(x) \geqslant F(u_0, v_0) = f(x_0).$$

This shows that x_0 will minimize $f(x)$. Again, as for any $(u, v) \in T$, we have

$$F(u, v) \leqslant f(x_0) = F(u_0, v_0),$$

therefore, (u_0, v_0) maximizes $F(u, v)$.

Lemma 2:

Let x_0 be a solution to the primal problem. Then x_0 will also be a solution to the following linear programming problem.

$$\begin{aligned} &\text{Minimize } [f(x_0)]'x \\ &\text{Subject to } x \in S \end{aligned} \tag{7}$$

Proof:

Let x_0 minimize (7). Then, we must have $z_j - c_j \leqslant 0$, for all j's $= 1$, 2, ..., n (Hadley, 1963). In the present context, optimality condition of linear programming reduces to

$$f_{x_j}(x_0) - Y_j' f_{x_B}(x_0) \geqslant 0$$

This is same as (2), the condition for x_0 to minimize the primal problem. This proves the lemma.

Theorem 3:

If x_0 solves the primal problem, then there exists v_0 so that (x_0, v_0) solves the dual problem, and the extrema are equal.

Proof:

From above lemma, we have seen that if x_0 minimizes (P-P), then x_0 is a solution of (7) also. Further, dual of linear programming problem (7) is to

$$\begin{aligned} &\text{Maximize } b'v \\ &\text{Subject to } A'v \leqslant f_x(x_0) \\ &\qquad v \geqslant 0 \end{aligned} \tag{8}$$

Duality in linear programming ensures existence of optimal v_0 for problem (8) and that the optimal values of objective functions in both the cases are equal i.e.

$$b'v_0 = [f_x(x_0)]' \cdot x_0 \tag{9}$$

From (8) and (9) it is clear that (x_0, v_0) is a feasible solution for the dual problem. The fact that (x_0, v_0) is optimal for the (D-P) is evident from Theorems 1 and 2. Moreover, optimal values of objective functions in both the cases are equal.

3. DUALITY IN LINEAR FRACTIONAL PROGRAMMING

Let our primal problem be

$$\begin{aligned} &\text{Minimize } f(x) = \frac{cx + c_0}{dx + d_0} = \frac{G(x)}{H(x)} \\ &\text{Subject to } Ax \geqslant b \\ &\qquad x \geqslant 0 \end{aligned} \tag{10}$$

Here c and d are row vectors with n components and c_0, d_0 are arbitrary constants. The objective function $f(x)$ is quasimonotonic in nature. The dual of (10) will be to

$$\begin{aligned} &\text{Maximize } F(u,v) = \frac{cu + c_0}{du + d_0} = \frac{G(u)}{H(u)} = f(u) \\ &\text{Subject to } A'v + \frac{d}{H(u)} f(u) \leqslant \frac{c}{H(u)} \\ &\text{and} -b'v + \frac{cu}{H(u)} \leqslant \frac{du}{H(u)} f(u) \\ &u, v \geqslant 0 \end{aligned} \tag{11}$$

In (11), let us take $f(u) = t$, $cu = G(u) - c_0$, and $du = H(u) - d_0$. Dual of (10) becomes

$$\begin{aligned} &\text{Maximize } t \\ &\text{Subject to } A'v + \frac{d}{H(u)} t \leqslant \frac{c}{H(u)} \\ &\text{and } -b'v + \frac{d_0}{H(u)} t \leqslant \frac{c_0}{H(u)} \\ &u, v \geqslant 0 \end{aligned} \tag{12}$$

(10) and (12) are respectively the primal and dual problems of Kaska (1969).

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