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BAYESIAN ESTIMATION OF THE 3-PARAMETER INVERSE GAUSSIAN DISTRIBUTION

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ABSTRACT

The three-parameter inverse Gaussian distribution is used as an alternative model for the three parameter lognormal, gamma and Weibull distributions for reliability problems. In this paper Bayes estimates of the parameters and reliability function of a three parameter inverse Gaussian distribution are obtained. Posterior variance estimates are compared with the variance of their maximum likelihood counterparts. Numerical examples are given.

Key words: Inverse Gaussian distribution, Jeffreys prior, posterior, squared error loss function, reliability function, approximate Bayes estimate.

1. INTRODUCTION

The three-parameter inverse Gaussian distribution with probability density function

$$f(x \mid \alpha, \mu, \lambda) = \{\lambda/[2\pi(x-\alpha)^3]\}^{1/2} \exp\{-\lambda(x-\alpha-\mu)^2/[2\mu^2(x-\alpha)]\} \quad (1)$$

$$x > \alpha \quad ; \quad \mu, \lambda > 0$$

and reliability function

$$R_t = \phi(t_1) - k\phi(t_2) \quad (2)$$

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where

$$t_1 = \sqrt{\lambda/(t-\alpha)}\,[1-(t-\alpha)/\mu], \quad t_2 = \sqrt{\lambda/(t-\alpha)}\,[1+(t-\alpha)/\mu],$$

$k = \exp(2\lambda/\mu)$, and $\phi(\cdot)$ is the *cdf* of the standard normal distribution, is a generalization of the well known two parameter inverse Gaussian distribution where the threshold parameter α is introduced to represent the unknown origin. Sampling theory estimation of the parameters in the three parameter inverse Gaussian distribution is considered by many authors. For example, moment and maximum likelihood estimators and their asymptotic efficiency are discussed by Padgett and Wei (1979), Cheng and Amin (1981) and Jones and Cheng (1984). Modified maximum likelihood and modified moment estimators are considered by Chan, Cohen and Whitten (1984). Maximum likelihood estimators and their asymptotic variances and covariances for censored samples are obtained by Mahmoud (1983-1984).

The estimation of the reliability function of the two parameter inverse Gaussian distribution was considered by Chhikara and Folks (1977) and Padgett (1979). Bayesian estimation of the parameters and reliability function of the two parameter inverse Gaussian distribution was investigated by Banarjee and Bhattacharyya (1979), Padgett (1981) and Howlader (1985). No corresponding results are available for the three parameter inverse Gaussian distribution. This paper gives Bayes estimates of the parameters and reliability function of a three parameter inverse Gaussian distribution, based on an approximation developed by Lindley (1980), using Jeffreys' non-informative joint prior and a squared error loss function. The application of this approximation depends on the existence of the MLE of the parameters. Another form of the non-informative prior distribution is also assumed and the approximate estimation procedure is applied based on this form and a squared error loss function. The results are compared with those obtained using Jeffreys' prior.

2. MAXIMUM LIKELIHOOD ESTIMATION OF THE PARAMETERS AND RELIABILITY FUNCTION

Padgett and Wei (1979) and Cheng and Amin (1981) derived the maximum likelihood estimators of the parameters α, μ and λ in (1) as follows:

For a random sample $\underset{\sim}{x} = (x_1, x_2, ..., x_n)$ of size n from (1), the likelihood function is

$$l(\alpha, \mu\, \lambda \mid \underset{\sim}{x}) =$$
$$= [\lambda/(2\pi)]^{n/2} \prod_{i=1}^{n} (x_i - \alpha)^{-3/2} \exp\{-(\lambda/2\mu^2) \sum_{i=1}^{n} (x_i - \alpha - \mu)^2/(x_i - \alpha)\} \tag{3}$$

taking the logarithm we get

$$\begin{aligned} L(\alpha, \mu, \lambda \mid \underset{\sim}{x}) &= \log l(\alpha, \mu, \lambda \mid \underset{\sim}{x}) \\ &= (n/2) \log [\lambda/(2\pi)] - (3/2) \sum_{i=1}^{n} \log (x_i - \alpha) \\ &- [\lambda/(2\mu^2)] \sum_{i=1}^{n} [(x_i - \alpha - \mu)^2/(x_i - \alpha)] \end{aligned} \tag{4}$$

For fixed α, the maximum of L can be written as

$$L^*(\alpha) = L(\alpha, \hat{\mu}(\alpha), \hat{\lambda}(\alpha) \mid \underset{\sim}{x})$$

where

$$\hat{\mu}(\alpha) = \bar{x} - \alpha \quad \text{and} \quad \hat{\lambda}(\alpha) = \left\{ n^{-1} \sum_{i=1}^{n} (x_i - \alpha)^{-1} - \hat{\mu}^{-1}(\alpha) \right\}^{-1}$$

The overall maximum likelihood estimators $\hat{\alpha}, \hat{\mu}$ and $\hat{\lambda}$ can be found by maximizing $L^*(\alpha)$ with respect to α.

The maximum likelihood estimate of R_t can be obtained, using the invariance property, in the form

$$\hat{R}_t = \phi(\hat{t}_1) - \hat{k}\phi(\hat{t}_2) \tag{5}$$

where

$\hat{t}_1 = \sqrt{\hat{\lambda}/(t - \hat{\alpha})}\, [1 - (t - \hat{\alpha})/\hat{\mu}]$, $\hat{t}_2 = \sqrt{\hat{\lambda}/(-\hat{\alpha})}\, [1 + (t - \hat{\alpha})/\hat{\mu}]$ and $\hat{k} = \exp(2\hat{\lambda}/\hat{\mu})$.

Padgett and Wei (1979) discussed the conditions under which the maximum likelihood estimators exist. Cheng and Amin (1981) have shown that $L^*(\alpha)$ is bounded and consequently, $L^*(\alpha)$ have a global maximum at a stationary point where $\partial L^*(\alpha)/\partial\alpha = 0$.

3. BAYESIAN ESTIMATION OF THE PARAMETERS AND RELIABILITY FUNCTION

The case where the lower bound α is known is discussed by Banarjee and Bhattcharayya (1979) in a Bayesian analysis of a two parameter inverse Gaussian distribution. They considered a different parametric form than (1). Padgett (1981) pointed out that the use of Jeffreys' non-informative joint prior

$$p(\mu, \lambda) \propto (\mu^3\lambda)^{-1/2} \tag{6}$$

leads to an intractable posterior for estimating R_t. Howlader (1985) using a method developed by Lindley (1980) gave an approximate Bayes estimate of R_t under the prior (6). In this paper we estimate the three parameters α, μ and λ in (1) and extend the results for estimating the reliability function to include the threshold parameter α under two forms of the joint prior distribution of the parameters. First, we consider Jeffreys' non-informative joint prior which is given by

$$p_J(\alpha, \mu, \lambda) \propto \sqrt{|I(\alpha, \mu, \lambda)|}$$

where

$$|I(\theta_1, \theta_2, \theta_3)| = -E|\partial^2 \log f(x \mid \theta_1, \theta_2, \theta_3)/\partial\theta_i\partial\theta_j|, \quad i, j = 1, 2, 3$$

$$= -E\begin{vmatrix} 3/2(x-\alpha)^2 - \lambda/(x-\alpha)^3 & \lambda/\mu^3 & \{1/\mu^2 - 1/(x-\alpha)^2\}/2 \\ -\lambda/\mu^3 & \lambda\{2-3(x-\alpha)/\mu\}/\mu^3 & (x-\alpha-\mu)/\mu^3 \\ \{1/\mu^2 - 1/(x-\alpha)^2\}/2 & (x-\alpha-\mu)/\mu^3 & -1/\lambda^2 \end{vmatrix}$$

$$= \begin{vmatrix} \lambda/\mu^3 + 9/2\mu^2 + 21(1/\mu\lambda + 1/\lambda^2)/2 & \lambda/\mu^3 & 3(\lambda+\mu)/2\mu\lambda^2 \\ \lambda/\mu^3 & \lambda/\mu^3 & 0 \\ 3(\lambda+\mu)/2\mu\lambda^2 & 0 & 1/2\lambda^2 \end{vmatrix}$$

$$= 3(\lambda + 4\mu)/4\mu^5\lambda^3$$

Thus

$$p_J(\alpha, \mu, \lambda) \propto \frac{\sqrt{1/\mu + 4/\lambda}}{\lambda\mu^2} \tag{7}$$

combining the likelihood function (3) with the prior (7), the joint posterior *pdf* of α, μ and λ is

$$\pi_J(\alpha, \mu, \lambda \mid \underset{\sim}{x}) = K_J\sqrt{\lambda + 4\mu}\,\lambda^{(n-3)/2}\mu^{-5/2}\prod_{i=1}^{n}(x_i - \alpha)^{-3/2}\cdot$$
$$\cdot\exp\{-[\lambda/(2\mu^2)]\sum_{i=1}^{n}[(x_i - \alpha - \mu)^2/(x_i - \alpha)]\}$$

where K_J is a normalizing constant given by

$$K_J^{-1} = \int_0^{\infty}\int_0^{\infty}\int_{-\infty}^{x_{(1)}}\sqrt{\lambda + 4\mu}\,\lambda^{(n-3)/2}\mu^{-5/2}\prod_{i=1}^{n}(x_i - \alpha)^{-3/2}\cdot\exp\{$$
$$-[\lambda/(2\mu^2)]\sum_{i=1}^{n}[(x_i - \alpha - \mu)^2/(x_i - \alpha)]\}\,d\alpha\,d\mu\,d\lambda.$$

Another appropriate non-informative joint prior for the three parameters can be found by following the procedure adopted by Sinha and Sloan (1988) in dealing with the three parameter Weibull distribution and by Lye, Sinha and Booy (1988) for the three parameter lognormal distribution.

Suppose we are in ignorance about the parameters α, μ and λ so that the non-informative prior $p_1(\mu, \lambda) \propto (\mu^3\lambda)^{-1/2}$ and $p_2(\alpha) = \text{const.}$, would be appropriate.

It is reasonable to believe that α is distributed independently of μ and λ, since any prior knowledge about α is not likely to be influenced by one's knowledge about the values of these parameters (see Box and Taio (1973)). So the joint prior distribution $p(\alpha, \mu, \lambda)$ may be written as

$$\begin{aligned} p(\alpha, \mu, \lambda) &= p_1(\mu, \lambda)p_2(\alpha \mid \mu, \lambda) \\ &= p_1(\mu, \lambda)p_2(\alpha) \\ &\propto (\mu^3\lambda)^{-1/2} \end{aligned} \tag{8}$$

The joint posterior of (α, μ, λ) is given by

$$\pi(\alpha, \mu, \lambda \mid \underset{\sim}{x}) = Kl(\alpha, \mu, \lambda \mid \underset{\sim}{x})p(\alpha, \mu, \lambda)$$

where K is a normalizing constant.

It is well known that by using a squared error loss function criterion, the Bayes estimator of an arbitrary function, $u(\theta)$, of the parameters is the expectation of that function. Thus we have for $\theta = (\alpha, \mu, \lambda)$ and the prior $p(\alpha, \mu, \lambda)$

$$E\{u(\theta) \mid \underset{\sim}{x})\} = \frac{\int\int\int u(\theta)\, l(\alpha, \mu, \lambda \mid \underset{\sim}{x}) p(\alpha, \mu, \lambda)\, d\alpha\, d\mu\, d\lambda}{\int\int\int l(\alpha, \mu, \lambda \mid \underset{\sim}{x}) p(\alpha, \mu, \lambda)\, d\alpha\, d\mu\, d\lambda} \tag{9}$$

Since the reliability function is a function of the parameters it is a parameter itself. Thus (9) is applied to R_t.

In this paper we calculate $E(\theta_i \mid \underset{\sim}{x})$ and $E(\theta_i^2 \mid \underset{\sim}{x})$ in order to find the posterior variance estimates given by

$$\mathrm{Var}(\theta_i \mid \underset{\sim}{x}) = E(\theta_i^2 \mid \underset{\sim}{x}) - E^2(\theta_i \mid \underset{\sim}{x}) \quad ; \quad i = 1, 2, 3$$

where, $\theta_1 = \alpha$, $\theta_2 = \mu$, $\theta_3 = \lambda$. Also, $E(R_t \mid \underset{\sim}{x})$ is obtained.

The use of numerical integration computer routines would be required, which may or may not converge for the given set of data $\underset{\sim}{x}$.

In the following section, Bayes estimators are approximated by an asymptotic expansion of the ratio of two integrals which is due to Lindley (1980).

4. BAYESIAN APPROXIMATION

Since the integrals in (9) do not seem to take a closed form, an approximation is needed. Lindley (1980) developed an asymptotic expansion for the evaluation of the ratio of integrals of the form

$$\int u(\theta) v(\theta) \exp\{L(\theta)\}\, d\theta \Big| \int v(\theta) \exp\{L(\theta)\}\, d\theta \tag{10}$$

where $\theta = (\theta_1, \theta_2, ..., \theta_m)$, $L(\theta)$ is the logarithm of the likelihood function, and $u(\theta)$ and $v(\theta)$ are arbitrary functions of θ.

If $v(\theta)$ is the prior density of θ, then (10) is the posterior expectation of $u(\theta)$. Thus (10) can be written as

$$u^* = E\{u(\theta) \mid \underset{\sim}{x}\} = \int u(\theta) \exp\{L(\theta) + \rho(\theta)\}\, d\theta / \int \exp\{L(\theta) + \rho(\theta)\}\, d\theta \tag{11}$$

where $\rho(\theta) = \log v(\theta)$.

Using Lindley's approximation, $E\{u(\theta) \mid \underset{\sim}{x}\}$ given by (11) may be asymptotically estimated by

$$u^* = E\{u(\theta) \mid \underset{\sim}{x}\} = \left[u + \frac{1}{2} \sum_i \sum_j (u_{ij} + 2u_i\rho_j)\sigma_{ij} + \frac{1}{2} \sum_i \sum_j \sum_k \sum_l L_{ijk}\sigma_{ij}\sigma_{kl}u_l \right]_{\hat{\theta}} + \text{terms of order } n^{-2} \text{ or smaller} \quad (12)$$

which is the Bayes estimator of $u(\theta)$ under a squared error loss function and where $i, j, k, l = 1, 2, \ldots, m$, $\hat{\theta} = \text{MLE}(\theta)$,

$$u = u(\theta), \quad u_i = \partial u/\partial\theta_i, \quad u_{ij} = \partial^2 u/\partial\theta_i\partial\theta_j, \quad L_{ijk} = \partial^3 L/\partial\theta_i\partial\theta_j\partial\theta_k,$$

$\rho_j = \partial\rho/\partial\theta_j$ and $\sigma_{ij} = (i, j)th$ element in the inverse of the matrix $\{-L_{ij}\}$ all evaluated at the MLE of the parameters. The method requires that $\hat{\theta}$ be the unique MLE of θ, although in most instances the local MLE produce acceptable estimates (see Sinha (1986)).

For $m = 3$, (12) reduces to

$$u^* = E(u \mid \underset{\sim}{x}) = u + (u_1a_1 + u_2a_2 + u_3a_3 + a_4 + a_5) + \frac{1}{2}[A(u_1\sigma_{11} + u_2\sigma_{12} + u_3\sigma_{13}) + B(u_1\sigma_{21} + u_2\sigma_{22} + u\sigma_{23}) + C(u_1\sigma_{31} + u_2\sigma_{32} + u_3\sigma_{33})] \quad (13)$$

evaluated at $\hat{\theta} = (\hat{\alpha}, \hat{\mu}, \hat{\lambda})$, where

$$a_i = \rho_1\sigma_{i1} + \rho_2\sigma_{i2} + \rho_3\sigma_{i3} \quad ; \quad i = 1, 2, 3$$

$$a_4 = u_{12}\sigma_{12} + u_{13}\sigma_{13} + u_{23}\sigma_{23}$$

$$\alpha_5 = \frac{1}{2}(u_{11}\sigma_{11} + u_{22}\sigma_{22} + u_{33}\sigma_{33})$$

$$A = \sigma_{11}L_{111} + 2\sigma_{12}L_{121} + 2\sigma_{13}L_{131} + 2\sigma_{23}L_{231} + \sigma_{22}L_{221} + \sigma_{33}L_{331}$$

$$B = \sigma_{11}L_{112} + 2\sigma_{12}L_{122} + 2\sigma_{13}L_{132} + 2\sigma_{23}L_{232} + \sigma_{22}L_{222} + \sigma_{33}L_{332}$$

$$C = \sigma_{11}L_{113} + 2\sigma_{12}L_{123} + 2\sigma_{13}L_{133} + 2\sigma_{23}L_{233} + \sigma_{22}L_{223} + \sigma_{33}L_{333}$$

and the subscripts 1, 2, 3 on the right-hand sides refer to α, μ, λ, respectively.

For the prior distribution (7) we have

$$\rho_J = \log p_J(\alpha, \mu, \lambda) = \text{constant} - 2\log\mu - \log\lambda + (1/2)\,log\,(1/\mu + 4/\lambda),$$

$$\hat{\rho}_{J1} = \partial\rho_J/\partial\alpha = 0, \quad \hat{\rho}_{J2} = \partial\rho_J/\partial\mu = -5/2\hat{\mu} + 2/(\hat{\lambda} + 4\hat{\mu})$$
$$\text{and} \quad \hat{\rho}_{J3} = \partial\rho_J/\partial\lambda = -3/2\hat{\lambda} + 1/2(\hat{\lambda} + 4\hat{\mu}).$$

For the prior distribution (8) we have

$$\rho = \log p(\alpha, \mu, \lambda) = \text{constant} - (3/2)\log\mu - (1/2)\log\lambda,$$

$$\hat{\rho}_1 = \partial\rho/\partial\alpha = 0, \quad \hat{\rho}_2 = \partial\rho/\partial\mu = -3/2\hat{\mu}, \quad \hat{\rho}_3 = \partial\rho/\partial\lambda = -1/2\hat{\lambda}.$$

The derived L_{ij}, $i, j = 1, 2, 3$ and L_{ijk}, $i, j, k = 1, 2, 3$ and the estimated variances and covariances of the MLE are given in appendix A.

For illustration we use the prior (8) to obtain Bayes estimates.

Let $u = \alpha$, $u_1 = 1$, $u_2 = u_3 = 0$, $u_{ij} = 0$, $i, j = 1, 2, 3$ then,

$$a_1 = -3\sigma_{12}/2\mu - \sigma_{13}/2\lambda, \quad a_2 = -3\sigma_{22}/2\mu - \sigma_{23}/2\lambda$$
$$a_3 = -3\sigma_{32}/2\mu - \sigma_{33}/2\lambda, \quad a_4 = a_5 = 0 \quad \text{and}$$

$$\alpha^* = E(\alpha \mid \underset{\sim}{x}) = \alpha - 3\sigma_{12}/2\mu - \sigma_{13}/2\lambda + \frac{1}{2}[\sigma_{11}A + \sigma_{21}B + \sigma_{31}C]$$

Similarly

$$\mu^* = E(\mu \mid \underset{\sim}{x}) = \mu - 3\sigma_{22}/2\mu - \sigma_{23}/2\lambda + \frac{1}{2}[\sigma_{12}A + \sigma_{22}B + \sigma_{32}C]$$

$$\lambda^* = E(\lambda \mid \underset{\sim}{x}) = \lambda - 3\sigma_{32}/2\mu - \sigma_{33}/2\lambda + \frac{1}{2}[\sigma_{13}A + \sigma_{23}B + \sigma_{33}C]$$

Also. Let $u = \alpha^2$. Then,

$$u = 2\alpha \;,\; u_{11} = 2 \;,\; u_2 = u_3 = 0 \;,\; u_{ij} = 0 \;;\; i, j = 1, 2, 3$$
$$a_4 = 0 \;,\; a_5 = \sigma_{11}$$

From (12), we have

$$E(\alpha^2 \mid \underset{\sim}{x}) = \alpha^2 - \alpha(3\sigma_{12}/\mu + \sigma_{13}/\lambda) + \sigma_{11} + \alpha(\sigma_{11}\text{A} + \sigma_{21}B + \sigma_{31}C)$$

Hence the posterior variances are given by

$$\begin{aligned}
\text{Var}(\alpha \mid \underset{\sim}{x}) &= E(\alpha^2 \mid \underset{\sim}{x}) - E^2(\alpha \mid \underset{\sim}{x}) \\
&= \hat{\sigma}_{11} - \left[\frac{1}{2}(3\sigma_{12}/\mu + \sigma_{13}/\lambda) - \frac{1}{2}(\sigma_{11}A + \sigma_{21}B + \sigma_{31}C)\right]^2_{\hat{\theta}} \\
&< \hat{\sigma}_{11} = \text{Var}(\hat{\alpha}) \\
\text{Var}(\mu \mid \underset{\sim}{x}) &= \hat{\sigma}_{22} - \left[\frac{1}{2}(3\sigma_{22}/\mu + \sigma_{23}/\lambda) - \frac{1}{2}(\sigma_{12}A + \sigma_{22}B + \sigma_{32}C)\right]^2_{\hat{\theta}} \\
&< \sigma_{22} = \text{Var}(\hat{\mu}) \\
\text{Var}(\lambda \mid \underset{\sim}{x}) &= \hat{\sigma}_{33} - \left[\frac{1}{2}(3\sigma_{32}/\mu + \sigma_{33}/\lambda) - \frac{1}{2}(\sigma_{13}A + \sigma_{23}B + \sigma_{33}C)\right]^2_{\hat{\theta}} \\
&< \sigma_{33} = \text{Var}(\hat{\lambda})
\end{aligned}$$

To estimate the reliability function R_t we substitute $u = R_t$ in (13), where R_t is given by (2).

On differentiating u with respect to the parameters α, μ and λ, the derived u_i and u_{ij} are given in appendix *B*.

An alternative approximation, to evaluate the ratio of integrals of the form (9), is developed by Tierney and Kadane (1986) as follows:

Let $l_1 = \{\log p(\theta) + L(\theta \mid \underset{\sim}{x})\}/n$ and $l_2 = \{\log u(\theta) + \log p(\theta) + L(\theta \mid \underset{\sim}{x})\}/n$

where $\theta = (\alpha, \mu, \lambda)$, then (9) can be written in the form

$$E\{u(\theta) \mid \underset{\sim}{x}\} = \frac{\int e^{nl_2}\, d\theta}{\int e^{nl_1}\, d\theta} \tag{14}$$

Tierney and Kadane (1986) expand each integral in (14) separately about the point which maximizes the integrand. Their method requires the evaluation of the first and second derivatives of the posterior density and equation (14), in the multiparameter case, takes the approximate value

$$E\{u(\theta) \mid \underset{\sim}{x}\} = \sqrt{\frac{|\Sigma_2|}{|\Sigma_1|}} \exp\{n[l_2(\hat{\theta}_2) - l_1(\hat{\theta}_1)]\}$$

where $\hat{\theta}_1$ and $\hat{\theta}_2$ maximize l_1 and l_2, respectively, and Σ_1 and Σ_2 are negatives of the inverse Hessians of l_1 and l_2 at $\hat{\theta}_1$ and $\hat{\theta}_2$, respectively.

For each function $u(\theta)$ of the parameters to be estimated, Tierney and Kadane method of approximation requires solving three nonlinear equations to find $\hat{\theta}_2$ and to calculate Σ_2. On the other hand, Lindley (1980) expands both the numerator and denominator of (14) about the MLE. His method requires finding the MLE of the parameters and the evaluation of the third derivatives of the likelihood function.

In the present case of the three parameter inverse Gaussian distribution, Lindley's approximation is simpler to apply since the MLE of the parameters can be easily found by solving a single equation as shown in section 2. The third derivative of the likelihood function is simple to calculate as given in appendix A.

In the following section the approximate Bayes estimates of the parameters and the reliability function using Lindley's approximation are found, for three sets of data, under Jeffreys' prior (7) as α_J^*, μ_J^*, λ_J^*, R_{Jt}^* and the assumed prior (8) as α^*, μ^*, λ^*, R_t^* and a squared error loss function and compared with the corresponding MLE $\hat{\alpha}$, $\hat{\mu}$, $\hat{\lambda}$ and $\hat{R}_t$.

5. NUMERICAL EXAMPLES

To illustrate the results given in this paper we use three real sets of data. The first two are successfully fitted by Cheng and Amin (1981) to the three parameter inverse Gaussian distribution. The third is quoted from Proschan (1963) and used by Howlader (1985) in an example on approximate Bayes estimation of reliability function of two parameter inverse Gaussian distribution. This set will be shown to fit satisfactorily to a three parameter inverse Gaussian distribution.

The first two sets are quoted from Dumonceaux and Antle (1973), showing 20 records on maximum flood levels and Engelhardt and Bain (1979) giving 10 ball-bearing lifetimes. These two sets are used by Cheng and Amin (1981) to compare the maximum likelihood estimation in the three parameter Weibull, lognormal and inverse Gaussian distributions. The MLE of the parameters as obtained by Cheng and Amin (1981) and the approximate Bayes estimates calculated from (13) under the two prior distributions (7) and (8) together with the variance of the MLE, $\hat{\sigma}_{ii} = \mathrm{Var}(\hat{\theta}_i)$, as calculated in appendix A and the posterior variance, $\mathrm{Var}(\theta_i \mid \underset{\sim}{x})$, are given in table I.

TABLE I

MLE and Bayes estimates of the Parameters

D & A ($n = 20$)			E & B ($n = 10$)		
$\hat{\alpha}$ 0.178	α_J^* 0.180	α^* 0.174	$\hat{\alpha}$ 139.7	α_J^* 138.9	α^* 136.4
$\hat{\mu}$ 0.245	μ_J^* 0.243	μ^* 0.249	$\hat{\mu}$ 80.8	μ_J^* 79.36	μ^* 83.0
$\hat{\lambda}$ 0.914	λ_J^* 0.963	λ^* 1.037	$\hat{\lambda}$ 88.2	λ_J^* 98.47	λ^* 116.6
$\mathrm{Var}(\hat{\alpha})$ 4.14×10^{-4}	$\mathrm{Var}_J(\alpha \mid \underset{\sim}{x})$ 4.10×10^{-4}	$\mathrm{Var}(\alpha \mid \underset{\sim}{x})$ 3.98×10^{-4}	$\mathrm{Var}(\hat{\alpha})$ 35.24	$Var_J(\alpha \mid \underset{\sim}{x})$ 34.66	$\mathrm{Var}(\alpha \mid \underset{\sim}{x})$ 24.54
$\mathrm{Var}(\hat{\mu})$ 4.67×10^{-4}	$\mathrm{Var}_J(\mu \mid \underset{\sim}{x})$ 4.63×10^{-4}	$\mathrm{Var}(\mu \mid \underset{\sim}{x})$ 4.51×10^{-4}	$\mathrm{Var}(\hat{\mu})$ 189.81	$\mathrm{Var}_J(\mu \mid \underset{\sim}{x})$ 187.73	$\mathrm{Var}(\mu \mid \underset{\sim}{x})$ 184.87
$\mathrm{Var}(\hat{\lambda})$ 0.0888	$\mathrm{Var}_J(\lambda \mid \underset{\sim}{x})$ 0.0863	$\mathrm{Var}(\lambda \mid \underset{\sim}{x})$ 0.0736	$\mathrm{Var}(\hat{\lambda})$ 1.793.7	$\mathrm{Var}_J(\lambda \mid \underset{\sim}{x})$ 1688.19	$\mathrm{Var}(\lambda \mid \underset{\sim}{x})$ 985.58

($\mathrm{Var}_J(\theta \mid \underset{\sim}{x})$ is the posterior variance based on Jeffreys' prior)

The MLE and Bayes estimates of the reliability function R_t, as calculated from (5) and (12), for different values of t for the two sets of data are tabulated in tables II and III respectively.

TABLE II

MLE and Bayes estimates of R_y (D & A data, n = 20)

t	0.26	0.28	0.30	0.32	0.34	0.36
$\hat{R}_t$	0.9796	0.9402	0.8793	0.8035	0.7204	0.6361
R_t^*	0.9810	0.9438	0.8847	0.8099	0.7269	0.6421
R_{Jt}^*	0.9569	0.8892	0.7952	0.6883	0.5796	0.4772
t	0.38	0.40	0.42	0.44	0.46	0.48
$\hat{R}_t$	0.5550	0.4800	0.4120	0.3518	0.2991	0.2535
R_t^*	0.5600	0.4836	0.4143	0.3528	0.2990	0.2524
R_{Jt}^*	0.3851	0.3054	0.2380	0.1824	0.1374	0.1012

TABLE III

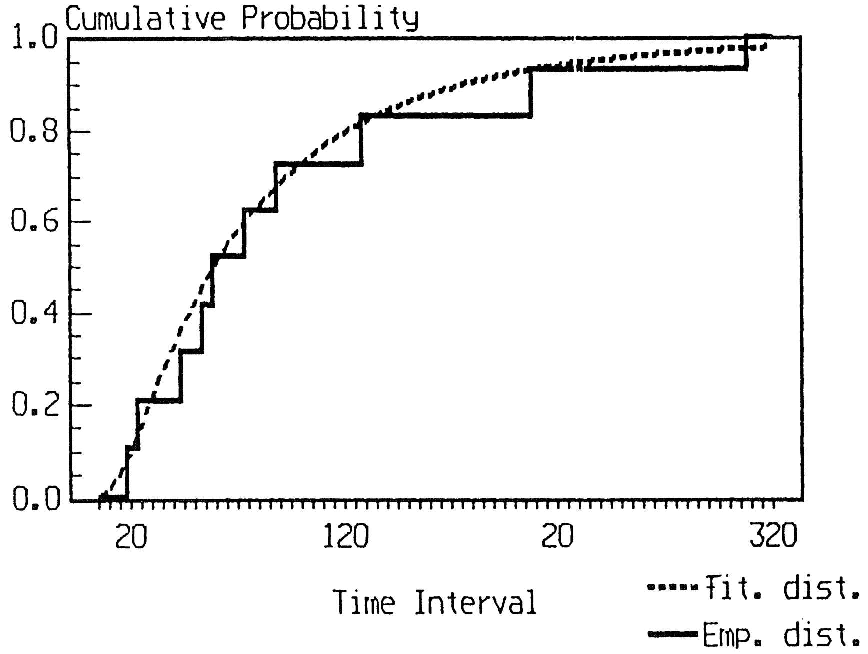
MLE and Bayes estimates of R_t (E & A data, n = 10)

t	150	160	170	180	190
$\hat{R}_t$	0.9904	0,9003	0.7726	0.6528	0.5509
R_t^*	0.9830	0.8972	0.7754	0.6583	0.556
R_{Jt}^*	0.9667	0.8351	0.6961	0.5789	0.4830
t	200	210	220	230	240
$\hat{R}_t$	0.4666	0.3972	0.3399	0.2924	0.2526
R_t^*	0.4698	0.3978	0.3381	0.2883	0.2469
R_{Jt}^*	0.4049	0.3412	0.2890	0.2460	0.2104

The third set of data is quoted from Proschan (1963). It consists of 29 records of time intervals between successive failures of air conditioning system of a Boing 720 jet plane number 7909. The observations are:

10 14 20 23 24 25 26 29 44 44
49 56 59 60 61 62 70 76 79 84
90 101 118 130 156 186 208 208 310

Figure I. Emprical and Fitted Distributions: Proschan Data



To give some idea about the goodness of fit, we plot the empirical *cdf* and the *cdf* of the fitted inverse Gaussian distribution in Fig. I. The estimates of the mean, variance and skewness given by $\hat{\alpha} + \hat{\mu}$, $\hat{\mu}^{e3}/\hat{\lambda}$ and $3(\hat{\mu}/\hat{\lambda})^{1/2}$ are compared with the corresponding sample statistics showing reasonable agreement. These values are

	α	mean	variance	skewness
Sample values	$(x_1 = 10)$	83.517	4840.6	1.52
Estimate	-3.523	83.517	6011.5	2.67

The MLE and Bayes estimates of the parameters and the variance of the MLE and the posterior variance are shown in table IV.

The MLE and Bayes estimates of the reliability function R_t for different values of t are tabulated in table V.

TABLE IV

MLE and Bayes estimates of the parameters (Proschan data, n=29)

$\hat{\alpha}$	α_J^*	α^*	$Var(\hat{\alpha})$	$Var_J(\alpha \mid \underset{\sim}{x})$	$Var(\alpha \mid \underset{\sim}{x})$
−3.523	−3.551	−4.504	15.496	15.495	14.535
$\hat{\mu}$	μ_J^*	μ^*	$Var(\hat{\mu})$	$Var_J(\mu \mid \underset{\sim}{x})$	$Var(\mu \mid \underset{\sim}{x})$
87.04	86.47	87.70	55.06	54.73	54.62
$\hat{\lambda}$	λ_J^*	λ^*	$Var(\hat{\lambda})$	$Var_J(\lambda \mid \underset{\sim}{x})$	$Var(\lambda \mid \underset{\sim}{x})$
109.69	112.68	119.66	870.07	861.16	770.74

TABLA V

MLE and Bayes estimates of R_t (Proschan data, n=29)

t	10	15	20	30	40	50
$\hat{R}_t$	0.9857	0.9526	0.9046	0.7910	0.6791	0.5801
R_t^*	0.9841	0.9517	0.9043	0.7917	0.6804	0.5813
R_{Jt}^*	0.9772	0.9373	0.8839	0.7651	0.6526	0.5547
t	60	70	80	90	100	110
$\hat{R}_t$	0.4956	0.4245	0.3648	0.3145	0.2722	0.2363
R_t^*	0.4964	0.4246	0.3642	0.3134	0.2706	0.2343
R_{Jt}^*	0.4720	0.4027	0.3448	0.2962	0.2554	0.2210

6. CONCLUSION

An approximation developed by Lindley (1980) is used to find Bayes estimates of the parameters and reliability function of a three parameter inverse Gaussian distribution. The formula is easy to use for any number of parmeters in the model. A simple program can be written on a PC microcomputer with no need for numerical integration. The method depends on the existence of the MLE of the parameters and provide estimates with smaller variance than the MLE.

APPENDIX A

From (4),

$$L_1 = \partial L/\partial\alpha = (3/2)\sum_{i=1}^{n}(x_i-\alpha)^{-1} + (n\lambda/2\mu^2) - (\lambda/2)\sum_{i=1}^{n}(x_i-\alpha)^{-2}$$

$$L_2 = \partial L/\partial\mu = (\lambda/\mu^3)\sum_{i=1}^{n}(x_i-\alpha-\mu)$$

$$L_3 = \partial L/\partial\lambda = (n/2\lambda) - \sum_{i=1}^{n}[(x_i-\alpha-\mu)^2/2\mu^2(x_i-\alpha)]$$

So that

$$\hat{L}_{11} = \partial^2 L/\partial\alpha^2 = (3/2)\sum_{i=1}^{n}(x_i-\hat{\alpha})^{-2} - \hat{\lambda}\sum_{i=1}^{n}(x_i-\hat{\alpha})^{-3}$$

$$\hat{L}_{12} = \partial^2 L/\partial\alpha\partial\mu = \hat{L}_{22} = \partial^2 L/\partial\mu^2 = -n\hat{\lambda}/\hat{\mu}^3,$$

$$\hat{L}_{13} = \partial^2 L/\partial\alpha\partial\lambda = (1/2)\left[n/\hat{\mu}^2 - \sum_{i=1}^{n}(x_i-\hat{\alpha})^{-2}\right],$$

$$\hat{L}_{23} = \partial^2 L/\partial\mu\partial\lambda = 0, \quad \hat{L}_{33} = \partial^2 L/\partial\lambda^2 = -n/2\hat{\lambda}^2,$$

$$\hat{L}_{111} = \partial^3 L/\partial\alpha^3 = 3\left[\sum_{i=1}^{n}(x_i-\hat{\alpha})^{-3} - \hat{\lambda}\sum_{i=1}^{n}(x_i-\hat{\alpha})^{-4}\right],$$

$$\hat{L}_{112} = \partial^3 L/\partial\alpha^2\partial\mu = 0 \quad , \quad \hat{L}_{113} = \partial^3 L/\partial\alpha^2\partial\lambda = -\sum_{i=1}^{n}(x_i-\hat{\alpha}^{-3}),$$

$$\hat{L}_{122} = \partial^3 L/\partial\alpha\partial\mu^2 = 3n\hat{\lambda}/\hat{\mu}^4 \quad ,\hat{L}_{123} = \partial^3 L/\partial\alpha\partial\mu\partial\lambda = -n/\hat{\mu}^3,$$

$$\hat{L}_{222} = \partial^3 L/\partial\mu^3 = 6n\hat{\lambda}/\hat{\mu}^4 \quad , \quad \hat{L}_{223} = \partial^3 L/\partial\mu^2\partial\lambda = -n/\hat{\mu}^3,$$

$$\hat{L}_{331} = \partial^3 L/\partial\alpha\partial\lambda^2 = \hat{L}_{233} = \partial^3 L/\partial\mu\partial\lambda^2 = 0 \quad , \quad \hat{L}_{333} = \partial^3 L/\partial\lambda^3 = n/\hat{\lambda}^3$$

The estimated variances and coveriances of the MLE are given as the elements in the inverse of the matrix $\{-L_{ij}\}$ all evaluated at the MLE.

Hence,

$$\hat{\sigma}_{11} = -\hat{\sigma}_{12} = n/(2\hat{\lambda}\hat{\mu}^3 D),$$

$$\hat{\sigma}_{13} = -\hat{\sigma}_{23} = -3n\sum_{i=1}^{n}(x_i - -\hat{\alpha})^{-1}/(2\hat{\mu}^3 D),$$

$$\hat{\sigma}_{22} = (1/2\hat{\lambda}^2 D)\left\{n\left[\hat{\lambda}\sum_{i=1}^{n}(x_i-\hat{\alpha})^{-3}-(3/2)\sum_{i=1}^{n}(x_i-\hat{\alpha})^{-2}\right]- -(9/2)\left[\sum_{i=1}^{n}(x_i-\hat{\alpha})^{-1}\right]^2\right\},$$

$$\hat{\sigma}_{33} = (n\hat{\lambda}/\hat{\mu}^3 D)\left[\hat{\lambda}\sum_{i=1}^{n}(x_i-\hat{\alpha})^{-3}-(3/2)\sum_{i=1}^{n}(x_i-\hat{\alpha})^{-2}-n\hat{\lambda}/\hat{\mu}^3\right],$$

where,

$$D = (n/2\hat{\lambda}\hat{\mu}^3)\left\{n\left[\hat{\lambda}\sum_{i=1}^{n}(x_i-\hat{\alpha})^{-3}-(3/2)\sum_{i=1}^{n}(x_i-\hat{\alpha})^{-2}-n\hat{\lambda}/\hat{\mu}^3\right] -(9/2)\left[\sum_{i=1}^{n}(x_i-\hat{\alpha})^{-1}\right]^2\right\}.$$

APPENDIX B

$$u = R_t = \phi(t_1) - k\phi(t_2)$$

where

$$t_1 = \sqrt{\lambda/(t-\alpha)}\,[1-(t-\alpha)/\mu], \quad t_2 = \sqrt{\lambda/(t-\alpha)}\,[1+(t-\alpha)/\mu],$$

$k = \exp(2\lambda/\mu)$, and $\phi(\cdot)$ is the *cdf* of the standard normal distribution.

$$u_1 = \partial u/\partial\alpha = [kt_1 f(t_2) - t_2 f(t_1)]/[2(t-\alpha)],$$

$$u_2 = \partial u/\partial\mu = \{\sqrt{\lambda(t-\alpha)}\,[f(t_1)-kf(t_2)] + 2k\lambda\phi(t_2)\}/\mu^2,$$

$$u_3 = \partial u/\partial\lambda = [t_1 f(t_1) - kt_2 f(t_2)]/2\lambda - 2k\phi(t_2)/\mu,$$

$$u_{11} = \partial^2 u/\partial\alpha^2 = \{(2+t_1t_2)[kt_1 f(t_2) - t_2 f(t_1)] - kt_2 f(t_2) + t_1 f(t_1)\}/[4(t-\alpha)^2],$$

$$u_{12} = \partial^2 u/\partial\alpha\partial\mu = \{\sqrt{\lambda(t-\alpha)}\,(1-t_1t_2)[kf(t_2)-f(t_1)] - 2\lambda kt_1 f(t_2)\}/[2\mu^2(t-\alpha)],$$

$$u_{13} = \partial^2 u/\partial\alpha\partial\lambda = [t_2(t_1^2-1)f(t_1) + kt_1(1-t_2^2)f(t_2)]/[2\lambda(t-\alpha)] + kt_1 f(t_2)/[\mu(t-\alpha)],$$

$$u_{22} = \partial^2 u/\partial\mu^2 = \{\lambda(t-\alpha)[kt_2 f(t_2) - t_1 f(t_1)] - 4\lambda k\phi(t_2)(\lambda+\mu) - 2\sqrt{\lambda(t-\alpha)}\,[2\lambda kf(t_2) + k\mu f(t_2) - \mu f(t_1)]\}/\mu,$$

$$u_{23} = \partial^2 u/\partial\mu\partial\lambda = \{\sqrt{(t-\alpha)/\lambda}[(1-t_1^2)f(t_1) + kf(t_2)(t_2^2 - 1 - 4\lambda/\mu)] + 2k[t_2 f(t_2) + 2(1+2\lambda/\mu)\phi(t_2)]\}/2\mu^2,$$

$$u_{33} = \partial^2 u/\partial\lambda^2 = (1/4\lambda^2)[kt_2(1+t_2^2)f(t_2) - t_1(1+t_1^2)f(t_1)] - (2k/\mu)[t_2 f(t_2)/\lambda + 2\phi(t_2)/\mu],$$

where $f(\cdot)$ is the *pdf* of the standard normal distribution.

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