TRABAJOS DE ESTADISTICA
Vol. 6, núm. 1, 1991, pp. 17 a 26

BAYESIAN RELIABILITY ANALYSIS WHEN MULTIPLE EARLY FAILURES MAY BE PRESENT

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SUMMARY

This paper discusses the Bayesian reliability analysis for an exponential failure model on the basis of some ordered observations when the first p observations may represent "early failures" or "outliers". The Bayes estimators of the mean life and reliability are obtained for the underlying parametric model referred to as the $SB(p)$ model under the assumption of the squared error loss function, the inverted gamma prior for the scale parameter and a generalized uniform prior for the nuisance parameter.

Key words: Exchangeable model; Gauss hypergeometric function; labelled slippage model; outlier; pretest estimator; reliability function; $SB(p)$ model; squared error loss function.

1. INTRODUCTION

In a paper of Bhattacharya and Singh (1979), the problem of classical reliability estimation based on a first few ordered observations from the exponential distribution for situations where the first failure occurs at a very early stage of a life test and is suspected to be an "early failure"

Recibido: Agosto 1989.
Revisado: Abril 1990.

or an "outlier", was studied under a suitable parametric model. The case of multiple outliers for this parametric outlier model (POM) was subsequently studied in the classical set-up by Bhattacharya and Singh (1986), wherein the editor of the journal in question referred to this POM as the "SB Model". This paper discusses the Bayesian analysis for this multiple outlier $SB(p)$ model, that is, when the first $p(\geqslant 1)$ ordered observations from the exponential failure model are suspected to be "early failures" or "outliers". Our analysis is carried out under the assumption of the usual squared error loss function and suitable prior densities for the parameters involved. It may be mentioned that in a recent paper, the Bayesian reliability analysis of the $SB(1)$ model was discussed by Bhattacharya and Singh (1988) under a different assumption for the joint prior density of the underlying parameters.

2. THE *SB(p)* MODEL

The $SB(p)$ model for p outliers or early failures to be considered here can be described as follows: For a preassigned r, let $x_1, x_2, ..., x_r$ be the first r ordered observations available with $(n - r)$ survivors at the termination of the life test based on n test units as in Epstein and Sobel (1953). We assume that for a given $p(1 \leqslant p < r)$, the first p failure times $x_1, x_2, ..., x_p$ can possibly be «early failures» or «outliers», that is, they come from a population specified by the *pdf*

$$f_{v,\theta}(x) = \frac{1}{v\theta} f\left(\frac{x}{v\theta}\right) \quad (0 < x < \infty;\ \theta > 0,\ 0 < v \leqslant 1), \tag{2.1}$$

while each of the remaining observations $x_i (i = p + 1, p + 2, ..., r)$ and also the $(n - r)$ survivors come from a popultion with the *pdf*

$$f_{1,\theta}(x) = \frac{1}{\theta} f\left(\frac{x}{\theta}\right) \quad (0 < x < \infty;\ \theta > 0), \tag{2.2}$$

where

$$f(x) = e^{-x} \quad (0 < x < \infty). \tag{2.3}$$

The case when $v = 1$ is referred to as the "homogeneous case", while the case when $0 < v < 1$ is referred to as the "nonhomogeneous case". In this latter case only, $x_1, x_2, ..., x_p$ are genuine early failures or outliers and then the constant v provides a measure of what is usually referred to as the "spuriosity coefficient". This POM for multiple early failures is somewhat akin to the "labelled slippage model" of Barnett and Lewis (1978) and the possible outliers are identified here as in the paper of Veale (1975). Our model is essentially different from the so-called "exchangeable model" of other authors working on the theory of outlying observations (Kale and Sinha, 1971; Sinha, 1973). The assumption of the exhangeable model, wherein each of the sample observation has an equal probability of being an outlier, seems to us to be unrealistic in so far as the discordant observations, if any, are more likely to be the extremely small or the large ones (cf. Barnett and Lewis, op. cit., p. 37). In the $SB(p)$ model being considered here, one can test whether the first p ordered observations $x_1, x_2, ..., x_p$ are outliers by testing a preliminary hypothesis

$$H : v = 1$$

against the alternative

$$H' : 0 < v < 1,$$

following an approach that is similar to the one usually adopted for statistical inferences based on incompletely specified models (see e.g. Bancroft, 1964). This approach is available in Bhattacharya and Singh (1979, 1986), wherein an UMP test of H based on the likelihood ratio is discussed and a pretest estimator of θ, depending upon the result of this preliminary test, is studied in detail. Here we shall consider a Bayesian approach for the estimation of θ, wherein the test of possible discordancy on the basis of the nuisance parameter v is dispensed with, and the problem of possible outliers is taken care of quite naturally in the Bayesian framework. The Bayesian statistical analysis is carried out here under the asumptions of the squared error loss function, a priori independence of v and θ, the inverted gamma prior *pdf* for θ and a generalized uniform prior density for the nuisance parameter v (cf. Bhattacharya, 1967). Thus, the prior densities for θ and v are respectively given by the expressions

$$g(\theta) = \frac{\mu^{\rho}}{\Gamma(\rho)} \theta^{-(\rho+1)} e^{-\mu\theta^{-1}} \quad (0 < \theta < \infty;\ \mu, \rho > 0), \tag{2.4}$$

and

$$g(v) = (1 - b)v^{-b} \quad (0 < v \leqslant 1;\ 0 \leqslant b < 1). \tag{2.5}$$

It may be observed from (2.5) that the density $g(v)$ decreases when v moves away from zero towards unity, so that the a priori probability that the observations with mean $v\theta$ are the smallest observations will be large. This fact is compatible with our model assumptions. Now we are ready to carry out the Bayesian analysis in the next section except that we need the likelihood function (LF) of the observed data under the $SB(p)$ model which is stated (cf. Bhattacharya and Singh, 1986, equation (3-1)) below:

$$l(\theta, v) = \frac{n!}{(n-r)!}\left\{\prod_{j=1}^{p} b_j\right\}\theta^{-r}e^{-S'\theta^{-1}} \quad (0 < \theta < \infty,\ 0 < v \leqslant 1), \tag{2.6}$$

where

$$S' = v^{-1}\left(\sum_{i=1}^{p} x_i\right) + \left(\sum_{i=p+1}^{r} x_i\right) + (n-r)x_r, \tag{2.7}$$

and

$$b_j = (n - j + 1)^{-1}[(n-p) + v^{-1}(p - j + 1)] \quad (j = 1, 2..., p). \tag{2.8}$$

3. THE BAYES ESTIMATOR OF THE MEAN LIFE

We shall first rewrite the LF at (2.6) in a form that is amenable to the Bayesian analysis. Thus, the kernel of the LF for the $SB(p)$ model is written as

$$l(\theta, v) \propto \left\{\prod_{j=1}^{p} C_j\right\}\theta^{-r}e^{-\theta^{-1}[v^{-1}O_p + T_p]} \quad (0 < \theta < \infty,\ 0 < v \leqslant 1), \tag{3.1}$$

where the constants $C_j (j = 1, 2, ..., p)$ and the statistics O_p and T_p are defined below:

$$C_j = (n-p)\left[1 + \frac{(p - j + 1)}{(n-p)v}\right] \quad (j = 1, 2, ..., p), \tag{3.2}$$

$$O_p = \sum_{i=1}^{p} x_i, \tag{3.3}$$

$$T_p = \left(\sum_{i=p+1}^{r} x_i \right) + (n - r)x_r. \tag{3.4}$$

The expression at (3.1) is further rewritten as

$$l(\theta, v) \propto (n - p)^p \left\{ \sum_{j=0}^{p} \frac{S_{j,p}}{(n-p)^j v^j} \right\} \theta^{-r} e^{-\theta^{-1}[v^{-1}O_p + T_p]}$$

$$(0 < \theta < \infty,\ 0 < v \leqslant 1), \tag{3.5}$$

where $S_{0,\,p} \equiv 1$, $S_{1,\,p} = \dfrac{p(p+1)}{2}$, and for $j = 2, \ldots, p$, $S_{j,\,p}$ is the sum of all possible products of j distinct natural numbers from the set $\mathscr{P} = \{1, 2, 3, \ldots, p\}$. Clearly, for $j = 1, 2, \ldots, p$, $S_{j,\,p}$ can easily be computed as the sum of $\binom{p}{j}$ terms, each term being a product of j distinct numbers from the set $\mathscr{P}$. It is this LF at (3.5), that is to be combined by using the Bayes theorem, with the joint prior density of (θ, v), which can be written, under the assumptions stated earlier, as follows:

$$g(\theta, v) \propto v^{-b} \theta^{-(\rho+1)} e^{-\mu\theta^{-1}} \quad (0 < \theta < \infty,\ 0 < v \leqslant 1). \tag{3.6}$$

From (3.5) and (3.6), we obtain the Bayesian joint posterior density of (θ, v) as

$$g^*(\theta, v) = \frac{h(\theta, v)}{\displaystyle\int_0^\infty I(\theta)\, d\theta} \quad (0 < \theta < \infty,\ 0 < v \leqslant 1), \tag{3.7}$$

where

$$I(\theta) = \int_0^1 h(\theta, v)\, dv \quad (0 < \theta < \infty), \tag{3.8}$$

and

$$h(\theta, v) = (n - p)^p \sum_{j=0}^{p} \frac{S_{j,\,p}}{(n-p)^j} \phi_{j,\,k}(\theta, v), \tag{3.9}$$

where

$$\phi_{j,\,k}(\theta, v) = v^{-(b+j)} \theta^{-(r+\rho+1)} e^{-\theta^{-1}[\mu + v^{-1}O_p + T_p]}. \tag{3.10}$$

Here both the functions $h(\theta, v)$ and $\phi_{j,\,k}(\theta, v)$ are defined over the region

$\{(\theta, v): 0 < \theta < \infty,\ 0 < v \leqslant 1\}$. From (3.7), the marginal posterior density of θ is obtained as

$$g^*(\theta) = \frac{I(\theta)}{\int_0^\infty I(\theta)\,d\theta} \quad (0 < \theta < \infty). \tag{3.11}$$

Now if we write

$$J_s = \int_0^\infty \theta^s I(\theta)\,d\theta \quad (s = 0, 1, 2),$$

then the Bayes estimator of θ obtained from (3.11), under the assumption of the squared error loss function, is given by

$$\hat{\theta} = \frac{J_1}{J_0}. \tag{3.12}$$

We now evaluate

$$I(\theta) = (n-p)^p \sum_{j=0}^{p} \frac{S_{j,p}}{(n-p)^j}\, \phi^*_{j,k}(\theta), \tag{3.13}$$

where

$$\phi^*_{j,k}(\theta) = \int_0^1 \phi_{j,k}(\theta, v)\,dv \quad (0 < \theta < \infty). \tag{3.14}$$

It can be easily seen that

$$\phi^*_{j,k}(\theta) = (O_p)^{-(b+j-1)}\theta^{-(r+\rho-b-j+2)}e^{-\theta^{-1}[\mu+T_p]}\,. \Gamma\left(b+j-1, \frac{O_p}{\theta}\right), \tag{3.15}$$

where

$$\Gamma(\alpha, y) = \int_y^\infty e^{-t}t^{\alpha-1} \quad (y > 0) \tag{3.16}$$

is the incomplete gamma function defined in Erdélyi, et al. (1953b, p. 133, formula 2). It may be pertinent to mention here that for $y > 0$, we do not need the usual restriction $\alpha > 0$, required for the convergence of the complete gamma integral.

From (3.13), we can evaluate for $s = 0, 1, 2$, the integrals

$$J_s = \int_0^\infty \theta^s I(\theta)\, d\theta = (n-p)^p \sum_{j=0}^{p} \frac{S_{j,p}}{(n-p)^j} I_{j,k}^{(s)}, \tag{3.17}$$

where

$$I_{j,k}^{(s)} = \int_0^\infty \theta^s \phi_{j,k}^*(\theta)\, d\theta$$

$$= (O_p)^{-(b+j-1)} \int_0^\infty \theta^{-(r+\rho-b-j-s+2)} e^{-\theta^{-1}[\mu+T_p]} \cdot \Gamma\left(b+j-1, \frac{O_p}{\theta}\right) d\theta, \tag{3.18}$$

on using (3.15). The last integral can be evaluated by using the Laplace transform (cf. Erdélyi, et al., 1953b, p. 138, fórmula 8) of the incomplete gamma function given below:

$$\int_0^\infty e^{-St} t^{\beta-1} \Gamma(\alpha, t)\, dt = \frac{\Gamma(\alpha+\beta)}{\beta(1+S)^{\alpha+\beta}} \cdot {}_2F_1\left[1, \alpha+\beta; \beta+1; \frac{S}{1+S}\right], \tag{3.19}$$

provided that the following conditions

$$\operatorname{Re}\beta > 0 \quad , \quad \operatorname{Re}(\alpha+\beta) > 0 \quad , \quad \operatorname{Re} S > -\frac{1}{2} \tag{3.20}$$

hold, where ${}_2F_1[.,.;.;.]$ denotes the Gauss hypergeometric function (cf. Erdélyi, et al., 1953a; Slater, 1960). The conditions stated by the equation (3.20) hold good for all the integrals evaluated in this paper by using the Laplace integral (3.19), if we assume $r + \rho > p + 2$, that is, if $r \geqslant p + 2$. Now from (3.18) and (3.19), we obtain:

$$I_{j,k}^{(s)} = \frac{\Gamma(r+\rho-s)}{(r+\rho-b-j-s+1)(\mu+O_p+T_p)^{r+\rho-s}} \cdot$$

$${}_2F_1[1, r+\rho-s;\ r+\rho-b-j-s+2;\ z], \tag{3.21}$$

where

$$z = \frac{\mu+T_p}{\mu+O_p+T_p}. \tag{3.22}$$

Hence, from (3.17), we get for $s = 0, 1, 2$, the following expression:

$$J_s = \frac{(n-p)^p \Gamma(r+\rho-s)}{(\mu + O_p + T_p)^{r+\rho-s}} \sum_{j=0}^{p} \frac{S_{j,\,p}}{(n-p)^j (j+\rho-b-j-s+1)} \cdot$$

$$_2F_1[1, r+\rho-s;\ r+\rho-b-j-s+2;\ z]. \tag{3.23}$$

Hence, the Bayes estimator $\hat{\theta}$ of the mean life can be obtained from (3.12), and the variance V of the posterior distribution (3.11) can be obtained as

$$V = \left(\frac{J_2}{J_0}\right) - \left(\frac{J_1}{J_0}\right)^2. \tag{3.24}$$

4. BAYESIAN REALIBILITY ESTIMATION

The reliability function for the exponential failure model (2.2) corresponding to a prescribed "mission time" $t(>0)$ is given by

$$R = R_\theta(t) = e^{-t\theta^{-1}}. \tag{4.1}$$

The estimation of this function is of considerable interest in the literature on reliability analysis (see e.g. Bhattacharya, 1967). Under the assumption of the squared error loss function, the Bayes estimator of reliability is simply the posterior expectation of (4.1) with respect to the density $g^*(\theta)$ given by (3.11). Hence, the Bayes estimator $\hat{R}$ of the reliability function is given by

$$\hat{R} = \frac{\int_0^\infty e^{-t\theta^{-1}} I(\theta)\, d\theta}{\int_0^\infty I(\theta)\, d\theta} \cdot \tag{4.2}$$

The numerator N of the above expression can be worked out by using the Laplace integral (3.19) as in the last section. Thus, we obtain:

$$N = \frac{(n-p)^p \Gamma(r+\rho)}{(\mu + O_p + T_p + t)^{r+\rho}} \sum_{j=0}^{p} \frac{S_{j,\,p}}{(n-p)^j (r+\rho-b-j+1)} \cdot$$

$$_2F_1[1,\ r+\rho;\ r+\rho-b-j+2;\ z'], \tag{4.3}$$

where

$$z' = \frac{\mu + T_p + t}{\mu + O_p + T_p + t}. \tag{4.4}$$

Since the denominator of (4.2) is simply J_0, which has already been evaluated, the Bayes estimator $\hat{R}$ can be computed. To compute the variance V^* of the posterior distribution of reliability, we shall need the following ratio of the integrals:

$$\int_0^\infty R^2 g^*(\theta)\, d\theta = \frac{\displaystyle\int_0^\infty e^{-2t\theta^{-1}} I(\theta)\, d\theta}{\displaystyle\int_0^\infty I(\theta)\, d\theta}. \tag{4.5}$$

The numerator N' in the last expression is again worked out by using (3.19). Thus, we obtain:

$$N' = \frac{(n-p)^p \Gamma(r+\rho)}{(\mu + O_p + T_p + 2t)^{r+\rho}} \sum_{j=0}^{p} \frac{S_{j,p}}{(n-p)^j (r + \rho - b - j + 1)} \cdot$$

$${}_2F_1[1, r+\rho; r+\rho-b-j+2; z''], \tag{4.6}$$

where

$$z'' = \frac{\mu + T_p + 2t}{\mu + O_p + T_p + 2t}. \tag{4.7}$$

Hence, the expressión (4.5) can be computed, from which we have to subtract $(\hat{R})^2$ in order to obtain the variance V^*.

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