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FACTORIAL STUDY OF A CERTAIN PARAMETRIC DISTRIBUTION

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ABSTRACT

The general theory of factorial analysis of continuous correspondance (FACC) is used to investigate the binary case of a continuous probability measure defined as:

$$\begin{aligned} T(x, y) &= ay^n + b \quad , \quad (x, y) \in D \;\&\; n \in N \\ &= 0 \qquad\quad\;\; , \quad \text{elsewhere} \end{aligned}$$

Where $n \geqslant 0$, a and b are the parameters of this distribution. While the Domain D is a variable trapezoidal inscribed in the unit square. The trapezoid depends on two parameters α and β.

This problem is solved. As special cases of our problem we obtaine a complete solution fot two of them which correspondt to a particular form of the correlation matrix in the discrete case.

Key words: Hypergeometric differential equation, Hypergeometric function, factorial analysis of continuous measure, spectral analysis, linear homogenous Fredholm integral equation, Eigenvalue problem.

1. INTRODUCTION

Up to 1971, the factorial analysis of correspondence has been built in the case where both correspondent sets I and J are finite.

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It was interesting to generalize the theory to the case where I and J are probability spaces for the following reasons:

— The generalization is carried out in a more natural frame.
— It is possible that the experience gives a discrete correspondence, while in reality the correspondence takes place between infinite sets.
— For a correspondence between finite sets with sufficiently high cardinal number it is possible to work out a simple and global continuous model.

These remarks have incited M. Naouri [8] to build the theory of factorial analysis for a binary continuous correspondence, viz in the case where I and J are probability spaces; to determine the factors and eigenvalues of continuous correspondence denoted by $T(x, y)$ and defined on $I \times J = [0, 1] \times [0, 1]$. Later on, this theory have been developed by J. Benzecri [1, 2, 3], K. Hamouda, A. Yehia [4, 5, 6, 7], A. Tharwat [9] and others.

The conditional probabilities $(T(x, y)/T_1(x), T(x, y)/T_2(y))$ are defined the transition probabilities P_1 and P_2 from I to J and from J to I.

We will denote by $d\mu_x$ and $d\mu_y$ the respective $T_1(x)$ and $T_2(y)$ represents the simple marginal probability.

Let us recall that each of the eigenfunctions of the pair $(f_i(x), g_j(y))$ is measurable and defined on the corresponding space I (respectively J) with probability measured $T_1(x)(T_2(y))$, belonging to $L^2(I)(L^2(J))$.

This pair is considered as a factor associated to the eigenvalue $\lambda_i \in \lambda$ (λ measures the variation or dispersion of the given cloud in the n dimensional space while λ_i is a part of this variation along an axe).

M. Naouri could prove that the factors are given by the following two linear, homogenous, Fredholn integral equations:

For all $x, z \in I \quad , \quad y, y' \in J$

$$\lambda' g_J(y) = \int_I \frac{T(x, y)}{T_1(x) T_2(y)} f_I(x)\, d\mu_x \tag{1}$$

$$\lambda' f_I(x) = \int_J \frac{T(x, y)}{T_1(x) T_2(y)} g_J(y)\, d\mu_y \tag{2}$$

By elimination of $g_J(y)$ between (1), (2), we get

$$\lambda'^2 f(x) = \int_I \left[\int_J \frac{T(z,y)T(x,y)}{T_1(z)T_2(y)} \, d\mu_y \right] f_I(x)\, d\mu_x \tag{3}$$

Equation (3) may be simplified to the following form

$$\lambda T_1(z) f(z) = Uf(z) = \int_I k(z,x) f(x)\, d\mu_x \tag{4}$$

The problem reduces to the spectral analysis of the positive self adjoint operator U with kernel,

$$k(z,x) = \int_J \frac{T(z,y)T(x,y)}{T_1(z)T_2(y)} \, d\mu_y \tag{5}$$

He could then show that the resolution for I and J are identical.

In analogy to the formula:

$$f_{ij} = f_i f_j \sum_{i,j} [\lambda(\phi)]^{1/2} \phi^i \phi^j$$

For the analysis of discrete correspondence, M. Naouri [8] could prove that by using the modified Hilbert Schmidt theorem, that:

$$T(x,y) = T_1(x)T_2(y) \sum_i [\lambda_i]^{1/2} f_i(x) g_j(y)$$

Where the f and g are the two systems of eigenfunctions of operators $(T \cdot T^*)$ and $(T^* \cdot T)$.

While in the case of a probabilistic correspondence between a finite family of probability spaces $\{E_q : q \in (1, Q)\}$ measured by $p_E(X_1, X_2, \dots, X_Q) \geqslant 0$ as their product $E = \Pi\{E_q : q \in (1, Q)\}$ [2], the sequence of functions $\{\phi_{Eq} : q \in (1, Q)\}$ each function being measurable and defined on E_q, on which there is given a measurable $P_{Eq} \geqslant 0$, belonging to $L^2(E_q)$. Considered as a factor relative to the eigenvalue $\lambda_i \in \lambda$, is given by the following system of Q linear homogenous integral equations of Fredholm type,

For all $1, 1' \in (1, Q) \quad Y_q \in E_q \;\&\; Y_{q'} \in E_{q'}$

$$\phi_{Eq}(y_q) = \sum_{q'=1} \left[\int_{Eq'} D_{q,q'}(y_q, y_{q'}) \phi_{Eq'}(y_{q'})\, d\mu_{q'} : q \neq q' \right]$$

Where the Q bounded kernels $D_{q,q'}(Y_q, Y_{q'})$ are given by

$$D_{q,q'}(Y_q, Y_{q'}) = 0 \quad , \quad q = q'$$
$$= \frac{P_{Eq\sqrt{}Eq'}(Y_q, Y_{q'})}{P_{Eq}(Y_q) \cdot P_{Eq'}(Y_{q'})} \quad , \quad q \neq q'$$

and $P_{Eq\sqrt{}Eq'}(Y_q, Y_{q'})$ being the binary marginal distribution on the face $E_q \times E_{q'}$ deduced by the projection of the measure P_E on that face.

We have also shown that each function of the sequence $\{\phi_q : q \in (1, Q)\}$ is centered on its corresponding interval.

In our present work we use the general theory of factorial analysis of continuous correspondence (F.A.C.C.) to exhibit the binary case of continuous probability measure defined by $T(x, y)$ as follow:

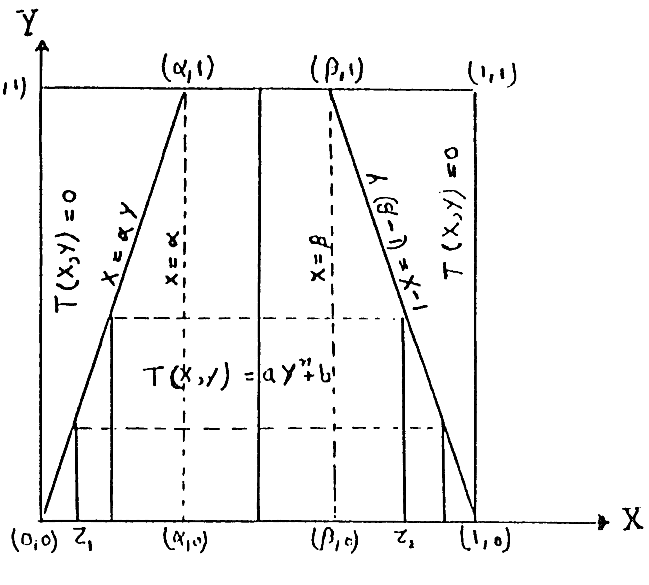
$$T(x, y) = ay^n + b \quad , \quad (x, y) \in D \quad , \quad n \in N$$
$$= 0 \quad , \quad \text{otherwise} \tag{6}$$

Where the domain D is the trapezoid inscribed in the square $[0, 1] \times [0, 1]$ and defined by:

$$0 \leqslant y \leqslant 1 \quad \text{and} \quad y \leqslant x/\alpha$$
$$y \leqslant (1 - x)/(1 - \beta) \quad \text{and} \quad 0 < \alpha \leqslant \beta < 1$$

and a, b are constants.

The simple graphical representation of this case is shown:



2. MAIN RESULTS

It can be shown that the two marginal probability measures $T_1(x)$ and $T_2(y)$ derived from the studies probability measure are given by:

$$
\begin{aligned}
T_1(x) &= a/(n+1)((X/\alpha)^{n+1} + b(x/\alpha)) &&, \quad x \leqslant \alpha \\
&= a/(n+1) + b &&, \quad \alpha \leqslant x \leqslant \beta \\
&= a/(n+1)[((1-x)/(1-\beta))^{n+1} + b(1-x)/(1-\beta)] &&, \quad \beta \leqslant x
\end{aligned}
$$

and

$$T_2(y) = (ay^n + b)(1 - cy) \quad , \quad 0 \leqslant y \leqslant 1$$

where

$$c = 1 + \alpha - \beta \tag{7}$$

Now, let us calculate the kernel $k(z, x)$ defined by (5) (we consider the cases where the integrand in (5) does not vanish).

It can be seen that $k(z, x)$ reduced to:

[i] if $z \leqslant \alpha$

$$
\begin{aligned}
k(z, x) &= R(x/\alpha) &&, \quad x \leqslant z \leqslant \alpha \\
&= R(z/\alpha) &&, \quad z \leqslant x \leqslant \alpha \\
& && \quad \text{or } \alpha \leqslant x \leqslant 1 - (z/\alpha)(1-\beta) \\
&= R((1-x)/(1-\beta) &&, \quad 1 - (z/\alpha)(1-\beta) \leqslant x
\end{aligned}
\tag{8.a}
$$

[ii] if $\alpha < z < \beta$

$$
\begin{aligned}
k(z, x) &= R(x/\alpha) &&, \quad x \leqslant \alpha \\
&= R(1) &&, \quad \alpha \leqslant x \leqslant \beta \\
&= R((1-x)/(1-\beta) &&, \quad \beta \leqslant x
\end{aligned}
\tag{8.b}
$$

[iii] if $\beta \leqslant z$

$$
\begin{aligned}
k(z, x) &= R(x/\alpha) &&, \quad x \leqslant \alpha(1-z)/(1-\beta) \\
&= R((1-z)/(1-\beta) &&, \quad \alpha(1-z)/(1-\beta) \leqslant x \leqslant \beta \\
& && \quad \text{or } \beta \leqslant x \leqslant z \\
&= R((1-x)/(1-\beta)) &&, \quad z \leqslant x
\end{aligned}
\tag{8.c}
$$

where:

$$R(s) = k_1 \sum_{r=1}^{n} (cs)^r(1/r) + k_2 \ln|1 - cs| \tag{9}$$

and

$$k_1 = -a/c^{n+1} \quad \& \quad k_2 = k_1 - b/c$$

It's clear that the constants a and b satisfy the following identity:

$$a\left[\frac{(n+2) - c(n+1)}{(n+1)(n+2)}\right] + b\frac{(2-c)}{2} = 1 \tag{10}$$

Noting that the kernel $k(z, x)$ is continuous, even differentiable to all orders in the different domains defined previously in this case(*).

In order to determine the factor characteraized by the eigenfunction $f(z)$ associated to the eigenvalue $\lambda_i \in \lambda$ for $0 < z \leqslant 1$, we use (7) and (8a, b, c) in (4). Hence,

[i] for $z \leqslant \alpha$

$$\lambda(a/(n+1)(z/\alpha)^{n+1} + b(z/\alpha))f(z) = \int_0^z R(x/\alpha)f(x)\,dx +$$
$$+ \int_z^{1-(z/\alpha)(1-\beta)} R(z/\alpha)f(x)\,dx + \int_{1-(z/\alpha)(1-\beta)}^{1} R((1-x)/(1-\beta))f(x)\,dx$$

Where $R(s)$ is given by (9).

Differentiating twice, we obtain:

$$\left[1 - \frac{cz}{\alpha}\right]\frac{\phi(z)}{\phi'(z)}f''(z) + \left[2 - \frac{cz}{\alpha} - \frac{c\phi(z)}{\alpha\phi'(z)} - \frac{cz\phi(z)\phi''(z)}{(\phi'(z))^2}\right]f'(z) +$$
$$+\left[\frac{1}{\lambda} - \frac{c}{\alpha}\right]f(z) + \frac{1-\beta}{\alpha}f\left[1 - \frac{z(1-\beta)}{\alpha}\right] = 0 \tag{11}$$

Where $\phi(z) = a/(n+1)(z/\alpha)^{n+1} + b(z/\alpha)$.

(*) The point $x = z = \alpha/c$ is singular (the point $c = 1$ i.e. $\alpha = \beta$ does not make any problems science if $\alpha = \beta$ the set $(z : \alpha < z < \beta)$ is the empty set), it will be excluded however; since the values of $k(z, x)$ on any set of measure zero may be discarded.

But from the centerlization property [3], we get:

$$\int_0^\alpha \phi(z)f(z)\,dz + \int_\alpha^\beta (a/(n+1)+b)f(z)\,dz +$$

$$+ \int_\beta^1 (\alpha(1-z)/(1-\beta))f(z)\,dz = 0 \tag{12}$$

By using the transformation $t = \alpha(1-z)/(1-\beta)$. In the third integral in equation (12) we get:

$$\int_\beta^1 \phi(\alpha(1-z)/(1-\beta))f(z)\,dz =$$

$$\int a/(n+1)\phi(t/\alpha)f(1-[(1-\beta)t/\alpha])(1-\beta)/\alpha\,dt \tag{*}$$

equations (5, 7) gives

$$f(z) = \text{constant} \quad ; \quad \alpha < z < \beta \tag{**}$$

From (*, **) we obtain

$$f((1-z)(1-\beta)/\alpha) = N/\phi(z) - (\alpha/(1-\beta))f(z) \tag{13}$$

Where

$$N = k(\beta-\alpha)/(1-\beta)((a/(n+1))+b)$$

and

$$k = \text{constant} = f(z) \quad , \quad \alpha \leqslant z \leqslant \beta$$

Using (13), equation (11) becomes:

$$\left[1-\frac{cz}{\alpha}\right]\frac{\phi(z)}{\phi'(z)}f''(z)+\left[2-\frac{cz}{\alpha}-\frac{c\phi(z)}{\alpha\phi'(z)}-\frac{cz\phi(z)\phi''(z)}{(\phi'(z))^2}\right]f'(z) -$$

$$- f'(z) - \frac{c}{\alpha}f(z) = N(1-\beta)/\alpha\phi(z) \tag{14}$$

[ii] For $\alpha < z < \beta$

$$\lambda(a/(n+1)+b)f(z) = \int_0^\alpha R(x/\alpha)f(x)\,dx +$$

$$\int_{\alpha}^{\beta} R(1)f(x)\,dx + \int_{\beta}^{1} R((1-x)/(1-\beta))f(x)\,dx$$

Differentiating once, we get:

$$(a/(n+1)+b)f'(z) = 0 \tag{15}$$

Which give

$$f(z) = \text{constant}$$

[iii] for $\beta \leqslant z \leqslant 1$

$$\lambda(a/(n+1)((1-z)/(1-\beta))^{n+1} + b((1-z)/(1-\beta)))f(z) =$$

$$= \int_{0}^{\alpha(1-z)/(1-\beta)} R(x/\alpha)f(x)\,dx + \int_{\alpha(1-z)/(1-\beta)}^{z} R((1-z)/(1-\beta))f(x)\,dx +$$

$$+ \int_{z}^{1} R((1-x)/(1-\beta))f(x)\,dx$$

Differentiating twice to get:

$$\left[1 - \frac{c(1-z)}{1-\beta}\right]\frac{\phi(\alpha(1-z)/(1-\beta))}{\phi'(\alpha(1-z)/(1-\beta)}f''(z) +$$

$$\left[1 - \frac{c(1-z)}{1-\beta}\right]\left[2 - \frac{\phi(\alpha(1-z)/(1-\beta))\phi''(\alpha(1-z)/(1-\beta))}{[\phi'(\alpha(1-z)/(1-\beta))]^2} +\right.$$

$$\left.\frac{c}{1-\beta}\left[\frac{\phi(\alpha(1-z)/(1-\beta))}{\phi'(\alpha(1-z)/(1-\beta))}\right]f'(z) + \left[\frac{1}{\lambda} - \frac{c}{1-\beta}\right]f(z) +\right.$$

$$+ \frac{\alpha}{1-\beta} f(\alpha(1-z)/(1-\beta)) = 0 \tag{16}$$

The substitution $t = \alpha(1-z)/(1-\beta)$ transforms equation (16) to equation (14). Hence the solution of equation (14) determines the function $f(z)$ which represents the factors in the intervals $0 < z \leqslant \alpha$ and $\beta \leqslant z \leqslant 1$ respectively and for the interval $\alpha < z < \beta$ it is constant.

Theorema (1):

Let $\{\phi_i(z)\}$be the eigenfunctions associated to an eigenvalue $\lambda_i \in \lambda$, defined on the unit interval in the plan for the continuous probability measure

$$T(x, t) = ay^n + b \quad , \quad (x, y) \in D \quad , \quad n \in N$$
$$= 0 \quad , \quad \text{otherwise}$$

Where the domain D is the trapezoid inscribed in the unit square given by:

$$0 \leqslant y \leqslant 1 \quad \text{and} \quad y \leqslant x/\alpha$$
$$y \leqslant (1 - x)/(1 - \beta) \quad \text{and} \quad 0 < \alpha \leqslant \beta < 1$$

and a, b are constants.

Then $\phi(z)$ is given by:

— $\phi(z)$ is constant for $\alpha \leqslant z \leqslant \beta$

for $0 < z < \alpha$

By using the transformation $z = \alpha(1 - t)/(1 - \beta)$for $\beta < z < 1$

Using the following relation

$$\lambda g(y) = \int_{E_x} \frac{T(x, y)}{T_1(x) T_2(y)} f(x)\, dx$$

We find that the determination of a factor on E_x (the eigenfunction $f(x)$) associated to an eigenvalue λ_i leads to the determination of a factor on E_y (the eigenfunction $g(y)$).

In the next part we obtain definite forms for the eigenfunction $f(z)$ in some important special cases.

3. SPECIAL CASES

The method and the program of the factorial analysis of correspondences between two sets I and J in discrete case (where $k(i, j)$ is the number of times that the element $i \in I$ associated to an element $j \in J$ and their matrix of correspondence in $k_{I \times J}$ are well known, Benzecri [2]) used in different domains of studies (sciences, economics, languiste, physics, medicine, etc).

In this section we discuses important situations where the matrix of correspondence has a variable trapezoidal form and for which the given continuous probability measure (6) can be used mathematically as a good approximation to the discrete case.

The following special cases will be considered almost in detail.

[a] $\alpha + \beta \neq 1$

[b] $\alpha + \beta = 1$

While the other parameters satisfy $n \geqslant 0$ and $b = 0$.

THE CASE [a]

This particular case corresponds to $n \geqslant 0$ & $b = 0$ and $\alpha + \beta \neq 1$, $0 < \alpha \leqslant \beta \leqslant 1$, if we put $b = 0$ in equations (5) to (14) and using the results of the preceding theorem we obtain:

[i] $$(a/(n+1))f'(z)=0 \quad \text{if } \& < z < \beta \tag{17}$$

and for all $0 < z \leqslant \alpha$ or $\beta \leqslant z \leqslant 1$ the D.E. (14) reduced to

[ii] $$z(1 - cz/\alpha)f''(z) + [(n+2) - (n+3)cz/\alpha f'(z) - \\ - (n+1)c/\alpha f(z) = k'((n+1) \tag{18}$$

where

$$k' = k(\beta - \alpha)/\lambda$$

and

$$k = f(z) = \text{constant} \quad , \quad z \in [\alpha, \beta]$$

To solve the differential equation (18), we obtain by the method of variation of parameters.

First, we find the solution of the homogeneous part of equation (18) which is a special case of Gauss' equation. The general form of Gauss' equation is:

$$S(1 - g)g''(s) + [h - (r + m + 1)s]g'(s) - rmg(s) = 0$$

Setting $s = x/m$, hence the previous D.E. of Gauss reduces to:

$$x(1 - x/m)g''(x) + [h - (1 + (r + 1/m))x]g'(x) - rg(x) = 0$$

and thus the homogeneous diferential equation (18) corresponds to the values of parameters:

$$r = 1 \quad \text{and} \quad m = n + 1 \quad \text{and} \quad h = n + 2$$

Hence the solution of the homogeneous part of (18) is given by

$$f(x) = AF(r, m, h, z/m) + BZ^{1-h}F(r + 1 - h, m + 1 - h, 2 - h, z/m)$$

i.e.

$$f(z) = AF(1, n + 1, n + 2, z/n + 1) + B/z^{n+1}$$

Where:

$$A, B \text{ are constants and } F(r, 0, h, z) = 0$$

Where the hypergeometric function F is given by:

$$F(r, m, h, x) = \Gamma(h)/[\Gamma(r)\Gamma(m)] \sum_{k=0}^{\infty} \Gamma(r + k)\Gamma(m + k)\Gamma(h + k)x^k/k!$$

To complete the solution of equation (18), we suggest that the general solution of it is given by

$$f(z) = A(z)f_1(z) + B(z)f_2(z) \tag{19}$$

Where:

$$f_1(z) = F(1, n + 1, n + 2, z/n + 1)\,\alpha f_2(z) = 1/z^{n+1} \tag{20}$$

and

$$A(z) = \int -g(z)f_2(z)/W\{f_1(z), f_2(z)\}\,dz$$

$$B(z) = \int g(z)f_1(z)/W\|f_1(z), f_2(z)\}\,dz$$

Where:

$$g(z) = k'(n + 1)/\lambda(\alpha/z)^{n+1}$$

$$w(f_1(z), f_2(z)) = \begin{vmatrix} f_1(z) & f_2(z) \\ f'_1(z) & f'_2(z) \end{vmatrix}$$

i.e.

$$W(f_1(z), f_2(z)) = -\lambda g(z)/k'\alpha^{n+1}[1/2F(1, n+1, n+2, 2/n+1) + \\ + (1/(n+2))F(2, n+2, n+3, z/n+1)] \qquad (21)$$

We find that the general solution of equation (18) is given by (19), where $f_1(z), f_2(z)$ are defined by (20) and

$$A(z) = \\ = \frac{\lambda}{k'\alpha^{n+1}} \int \frac{dz}{z^n F(1, n+1, n+2, z/(n+1)) + z(n+2)F(2, n+2, n+3, z/(n+1)}$$

$$B(z) = \\ = \frac{-\lambda}{k'\alpha^{n+1}} \int \frac{F(1, n+1, n+2, z/n+1)\, dz}{(1/2)F(1, n+1, n+2, z/(n+1)) + (1/(n+2))F(2, n+2, n+3, z/(n+1)}$$

THE CASE [b]

This particular case corresponds to $n \geqslant 0$ & $b = 0$ and $\alpha + \beta = 1$ $(0 < \alpha \leqslant \beta \leqslant 1)$ i.e. $\beta = 1 - \alpha$, from symmetry, we have $f(z) = f(1-z)$.

Introducing the values of the parameters in the D.E. (12) we obtain:

[i] $$f(z) = \text{constant} \quad \text{for } \alpha < z < 1 - \alpha \qquad (22)$$

while for $0 < z < \alpha$

[ii] $$z(1-2z)f''(z) + [(n+2) - 2(n+3)(1-z)]f'(z) + \\ + (n+1)((1/\lambda) - z)f(z) + ((n+1)/\lambda)f(1-z) = 0 \qquad (23)$$

and for $1 - \alpha \leqslant z \leqslant 1$

[iii] $$(1-z)(2z-1)f''(z) + [-(n+2) + 2(n+3)(1-z)]f'(z) + \\ + (n+1)((1/\lambda) - z)f(z) + ((n+1)/\lambda)f(1-z) = 0 \qquad (24)$$

Applying the result of symmetry, the D.E. (23) reduces to:

$$z(1-2z)f''(z) + [(n+2) - 2(n+3)z]f'(z) + 2(n+1)(1/\lambda - 1)f(z) = 0 \quad (25)$$

Equation (25) is a special case of Gauss' equation with the values of parameters:

$$r = n + 3/2 \quad \text{and} \quad m = 1/2 \quad \text{and} \quad h = n + 2$$

Hence the solution of equation (25) is of the form:

$$f(z) = AF(n + 3/2, 1/2, n + 2, 2z) + B(1/z^{n+1})F(1/2, -n - 1/2, -n, z) \quad (26)$$

Finally, the substitution $z = 1 - t$ transforms equation (24) to equation (23) hence the solution of equation (24) is given by equation (26).

Consiquently, the following theorem holds:

Theorem (2):

The regular solution of the continuous probability measure defined by $T(x, y) = ay^n$ on the trapezoid inscribed in the unit square by F.A.C.C. gives proper function (defining the factors) in terms of hypergeometric function defined in equation (19) where $\alpha + \beta \neq 1$. While for $\alpha + \beta = 1$, the proper function given also in terms of hypergeometric function defined in equation (26) for $0 < z \leqslant \alpha$ and $\beta \leqslant z \leqslant 1$ and gives a constant for $\alpha < z < \beta$.

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