TRABAJOS DE ESTADISTICA
Vol. 5, núm. 2, 1990, pp. 83 a 94

A CRAMER-RAO ANALOGUE FOR MEDIAN-UNBIASED ESTIMATORS

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SUMMARY

Adopting a measure of dispersion proposed by Alamo [1964], and extending the analysis in Stangenhaus [1977] and Stangenhaus and David [1978b], an analogue of the classical Cramér-Rao lower bound for median-unbiased estimators is developed for absolutely continuous distributions with a single parameter, in which mean-unbiasedness, the Fisher information, and the variance are replaced by median-unbiasedness, the first absolute moment of the sample score, and the reciprocal of twice the median-unbiased estimator's density height evaluated at its median point. We exhibit location-parameter and scale-parameter families for which there exist median-unbiased estimators meeting the bound. We also give an analogue of the Chapman-Robbins inequality which is free from regularity conditions.

Key words: median-unbiased estimator; Cramér-Rao; location; scale; diffusivity.

AMS 1970 *subject classification:* Primary 62F10.

1. INTRODUCTION

Let μ be the Lebesgue measure on Euclidean n-space E^n. Let $\mathscr{P} = \{P_\theta : \theta \in \Theta\}$ be the family of distribution functions which is absolutely

Recibido enero 1990.
Revisado abril 1990.
[1] Sung's work was supported in part by Non-directed Research Fund, Korea Research Foundation, 1990.

continuous with respect to μ and depends upon a single parameter θ, where the parameter space Θ is either the real line, or an interval on the real line. It is assumed that for every $\theta \in \Theta$, $f(x;\theta)$ is a continuous density function of P_θ with respect to μ, where $x \in E^n$.

Let $X = (X_1, X_2, ..., X_n)$ be a random vector of n iid random variables having a joint density function $f(x;\theta)$. Let $\tau(\theta)$ be a real-valued differentiable function on Θ. An estimator $\delta(X)$ of $\tau(\theta)$ is called *median-unbiased* if

$$\text{median}_\theta\, \delta(X) = \tau(\theta) \quad \text{for all} \quad \theta \in \Theta. \tag{1}$$

The usual *unbiasedness* will be called *mean-unbiasedness* to avoid ambiguity in the present discussion. For any estimator with absolutely continuous distribution, the condition (1) of median-unbiasedness reduces to

$$P_\theta[\delta(X) \leqslant \tau(\theta)] = P_\theta[\delta(X) \geqslant \tau(\theta)] = 1/2 \quad \text{for} \quad \text{all} \quad \theta \in \Theta.$$

Let $Y \equiv \delta(X)$ be any median-unbiased estimator of $\tau(\theta)$. Let $g_\delta(y;\theta)$ be a known continuous density function of Y.

An analogue of the Cramér-Rao lower bound for median-unbiased estimators was first proposed by Alamo [1964]. Since $E_\theta|\delta(X) - u|$ is minimized when u is a median of $\delta(X)$, Alamo postulated the mean absolute deviation as the measure of dispersion associated with median-unbiased estimators. Searching for a lower bound for the mean absolute deviation of median-unbiased estimators, Alamo found that such a lower bound is unobtainable independently of the estimator $\delta(X)$. Alamo proposed then as a new measure of dispersion associated with median-unbiased estimators the reciprocal of the squared density height of the median-unbiased estimator evaluated at its median $\tau(\theta)$, and developed an analogue of the Cramér-Rao lower bound for median-unbiased estimators, based on the expected squared sample score which also underlies the Cramér-Rao inequality. Alamo obtained, under certain regularity conditions,

$$1/\{g_\delta(\tau(\theta);\theta)\}^2 \geqslant 4[\tau'(\theta)]^2/I_2(\theta), \tag{2}$$

or equivalently,

$$1/2g_\delta(\tau(\theta);\theta) \geqslant |\tau'(\theta)|/\sqrt{I_2(\theta)}, \tag{3}$$

where $I_2(\theta)$ is the usual Fisher information function:

$$I_2(\theta) = E_\theta[(\partial/\partial\theta)\log f(x;\theta)]^2.$$

Though (2) is the original form of Alamo's inequality and the left-hand side of (2) was asserted as a measure of dispersion for median-unbiased estimators, we slightly change the form of the inequality to (3) to conform to the rest of this paper. Since it is desirable, in general, to have an estimator of which the density is concentrated around the parametric function of interest, it is not surprising that $1/2g_\delta(\tau(\theta);\theta)$ can be considered as a measure of dispersion.

The class of estimators that attains the lower bound, however, seems to be very restricted. An example where the lower bound is achieved, as was shown by Alamo, is that of estimating the location parameter from a double exponential distribution with a single observation.

A sharper lower bound than that of Alamo was proposed by Stangenhaus [1977], and Stangenhaus and David [1978a], in developing a lower bound for an estimator δ which minimizes $E_\theta|\delta(X) - \tau(\theta)|^\gamma$, where $\gamma > 1$. Using the fact that as $\gamma \downarrow 1$, an estimator which minimizes $E_\theta|\delta(X) - \tau(\theta)|^\gamma$ becomes a median-unbiased estimator, Stangenhaus proposed an analogue of the Cramér-Rao inequality based on the expected absolute sample score under considerably restrictive regularity conditions:

$$1/2g_\delta(\tau(\theta);\theta) \geqslant |\tau'(\theta)|/I_1(\theta), \tag{4}$$

where I_1 is an analogue of Fisher information:

$$I_1(\theta) = E_\theta|(\partial/\partial\theta)\log f(x;\theta)|.$$

Stangenhaus and David [1978b] rederived this inequality using a direct approach not involving the γ-limiting procedure under relaxed regularity conditions only for the case of $\tau(\theta) = \theta$ when the density of a median-unbiased estimator is defined over the entire real line:

$$1/2g_\delta(\theta;\theta) \geqslant 1/I_1(\theta).$$

They also identified bound-achieving median-unbiased estimators for symmetric continuous density functions defined over the entire real line which belong to a location family.

In this paper we extend these considerations to the treatment of arbitrary parametric function $\tau(\theta)$ of θ and give location-scale parameter families for which there exist median-unbiased estimators meeting the lower bound without assuming the symmetry condition.

It might be observed that the asymptotic standard deviation of the sample median from a density with θ as its location parameter is proportional to the reciprocal of the density height evaluated at θ, and the translated sample median can be taken as a median-unbiased estimator of the location parameter. Obviously, the asymptotic standard deviation of such a translated sample median is again proportional to the reciprocal of the density height at θ and the reciprocal of the density height of the translated sample median at θ is also proportional to the asymptotic standard deviation. This result sways us to the possibility of using estimator's density height as a natural measure of dispersion for median-unbiased estimators.

We call the common left-hand side term in (3) and (4) *difussivity*. Diffusivity is different from the conventional measure of dispersion in that it measures vertical spread of a density rather than horizontal spread.

2. LOWER BOUND

We first derive an analogue of the Chapman-Robbins inequality [1951] which is free from regularity conditions. Let $\Delta\theta$ be positive such that both θ and $\theta + \Delta\theta$ belong to Θ. By the definition of the median-unbiased estimator, the followings hold:

$$\int_{-\infty}^{\tau(\theta)} g_\delta(y;\theta)\,dy = \int_{-\infty}^{\tau(\theta+\Delta\theta)} g_\delta(y;\theta+\Delta\theta)\,dy = 1/2 \quad \text{for all} \quad \theta\in\Theta, \quad (5)$$

and

$$\int_{\tau(\theta)}^{+\infty} g_\delta(y;\theta)\,dy = \int_{\tau(\theta+\Delta\theta)}^{+\infty} g_\delta(y;\theta+\Delta\theta)\,dy = 1/2 \quad \text{for all} \quad \theta\in\Theta, \quad (6)$$

when θ and $\theta + \Delta\theta$ belong to Θ.

Consider $P_{\theta+\Delta\theta}[\tau(\theta) < Y \leqslant \tau(\theta + \Delta\theta)]$. We write this probability as follows, using (5):

$$\int_{\tau(\theta)}^{\tau(\theta+\Delta\theta)} g_\delta(y;\theta)+\Delta\theta)\,dy$$

$$= \int_{-\infty}^{\tau(\theta + \Delta\theta)} g_\delta(y; \theta + \Delta\theta)\, dy - \int_{-\infty}^{\tau(\theta)} g_\delta(y; \theta + \Delta\theta)\, dy$$

$$= \int_{-\infty}^{\tau(\theta)} g_\delta(y; \theta)\, dy - \int_{-\infty}^{\tau(\theta)} g_\delta(y; \theta + \Delta\theta)\, dy$$

$$= \int_{[x : \delta(x) \leqslant \tau(\theta)]} \{f(x; \theta) - f(x; \theta + \Delta\theta)\} \mu(dx). \tag{7}$$

An analogous argument for $[x : \delta(x) > \tau(\theta)]$, using (6), leads to

$$\int_{\tau(\theta)}^{\tau(\theta + \Delta\theta)} g_\delta(y; \theta + \Delta\theta)\, dy$$

$$= \int_{[x : \delta(x) > \tau(\theta)]} \{f(x; \theta + \Delta\theta) - f(x; \theta)\} \mu(dx). \tag{8}$$

Note that by the mean value theorem the commont left-hand side term of (7) and (8) can be written as

$$\int_{\tau(\theta)}^{\tau(\theta + \Delta\theta)} g_\delta(y; \theta + \Delta\theta)\, dy$$

$$= [\tau(\theta + \Delta\theta) - \tau(\theta)] g_\delta(\tau(\theta) + \lambda(\tau(\theta + \Delta\theta) - \tau(\theta)); \theta + \Delta\theta), \tag{9}$$

for some λ such that $0 \leqslant \lambda \leqslant 1$.

After replacing the left-hand sides of (7) and (8) by (9), we divide them by $\Delta\theta$, take absolute values to both sides of (7) and (8), and add the resulting two equalities to obtain

$$2 \left| \frac{\tau(\theta + \Delta\theta) - \tau(\theta)}{\Delta\theta} \right| g_\delta(\tau(\theta) + \lambda(\tau(\theta + \Delta\theta) - \tau(\theta)); \theta + \Delta\theta)$$

$$\leqslant \int \left| \frac{f(x; \theta + \Delta\theta) - f(x; \theta)}{\Delta\theta} \right| \mu(dx). \tag{10}$$

(10) can be shown to be valid for negative $\Delta\theta$ such that $\theta + \Delta\theta \in \Theta$. Since we assumed that g_δ is continuous and $\tau(\theta)$ is its median, we conclude by taking the limit infimum as $\Delta\theta \to 0$ in (10)

$$2|\tau'(\theta)| g_\delta(\tau(\theta); \theta) \leqslant \liminf_{\Delta\theta \to 0} \int \left| \frac{f(x; \theta + \Delta\theta) - f(x; \theta)}{\Delta\theta f(x; \theta)} \right| f(x; \theta) \mu(dx).$$

Hence we proved the following theorem.

Theorem 1. Let $\tau(\theta)$ be a real valued differentiable function on Θ. Let $\delta(X)$ be a median-unbiased estimator having a continuous density g_δ. Then

$$\frac{1}{2g_\delta(\tau(\theta);\theta)} \geqslant \frac{|\tau'(\theta)|}{\liminf_{\Delta\theta\to 0} \int \left|\frac{f(x;\theta+\Delta\theta)-f(x;\theta)}{\Delta\theta f(x;\theta)}\right| f(x;\theta)\mu(dx)}, \tag{11}$$

where the limit infimum is taken over all $\Delta\theta \neq 0$.

Example 1. Let $X_1, X_2, \ldots, X_n$ be iid random variables from the uniform distribution with marginal density

$$f(x;\theta) = \begin{cases} 1 & \text{for} \quad \theta - 1/2 \leqslant x \leqslant \theta + 1/2, \\ 0 & \text{otherwise.} \end{cases}$$

Let $\delta(X)$ be the sample mid-range. Then the density function of δ is given by

$$g_\delta(y;\theta) = \begin{cases} n(1-2|y-\theta|)^{n-1} & \text{for} \quad \theta - 1/2 \leqslant y \leqslant \theta + 1/2, \\ 0 & \text{otherwise.} \end{cases}$$

Therefore δ is median-unbiased for θ, and $1/2g_\delta(\theta;\theta) = 1/2n$. If we write the integrand in Theorem 1 as $|\Delta\theta|^{-1}|f(x;\theta+\Delta\theta) - f(x;\theta)|$, then the integral is nothing but the content of the symmetric difference of two unit n-cubes displaced with respect to each other by an amount $\sqrt{n}\Delta\theta$ along the equiangular line. This content is $2n\Delta\theta$ to order $\Delta\theta$, so that the sample mid-range achieves the lower bound. □

It is possible that the denominator of the right hand side of (11) is ∞ for any $\Delta\theta$. In this case, (11) is still a valid, though trivial, inequality.

We now impose the following regularity conditions and present an analogue of the Cramér-Rao inequality.

i) Θ is either the real line, or an interval on the real line.
ii) $(\partial/\partial\theta)f(x;\theta)$ exists for every $\theta\in\Theta$. (12)
iii) $0 < E_\theta|(\partial/\partial\theta)\log f(x;\theta)| < \infty$ for every $\theta\in\Theta$.

Theorem 2. Let $\tau(\theta)$ be a real valued differentiable function on Θ.

Let $\delta(X)$ be a median-unbiased estimator having a continuous density g_δ. Then, under the regularity conditions (12),

$$1/(2g_\delta(\tau(\theta);\theta)) \geqslant |\tau'(\theta)|/I_1(\theta). \tag{13}$$

Proof: By ii) and iii) in (12), the denominator of the right hand side in (11) is bounded below by $\int |(\partial/\partial\theta)\log f(x;\theta)| f(x;\theta)\,\mu(dx)$, which is $I_1(\theta)$. □

Example 2. Let $X_1, X_2, \ldots, X_n$ be a sample of n independent observations from $N(\mu, \sigma^2)$, where σ^2 is known. The sample mean $\bar{X}$ is median-unbiased for μ since $\bar{X} \sim N(\mu, \sigma^2/n)$. Therefore, the diffusivity of $\bar{X}$ is $\sigma\sqrt{\pi/2n}$. Since $T \equiv (\partial/\partial\mu)\log f(X;\mu) = \Sigma\,(X_i - \mu)/\sigma^2 \sim N(0, n/\sigma^2)$, then $I_1(\mu) = E_\mu|T| = \sqrt{2n/\pi\sigma^2}$. Hence, the sample mean attains the lower bound. Let M be the sample median. Asymptotically $M \sim N(\mu, 1/(4nf^2(\mu)))$. Hence M is median-unbiased for μ, and the diffusivity of M is $\pi\sigma/2\sqrt{n}$, which is greater than $1/I_1(\mu)$. Therefore the sample median is less efficient than the sample mean in the normal distribution. If we define the asymptotic relative efficiency between two median-unbiased estimators $\delta_1(X)$ and $\delta_2(X)$ of $\tau(\theta)$ to be the limit of the ratio $g_{\delta_1}(\tau(\theta))/g_{\delta_2}(\tau(\theta))$ as $n \to \infty$, then the asymptotic relative efficiency of $\bar{X}$ to M in the normal distribution has the value of $\sqrt{\pi/2} = 1.2533$. Note that we obtain the same value in mean-unbiased estimation, where the measure of relative efficiency is the ratio of the standard deviations. □

Example 3. Let $X_1, X_2, \ldots, X_n$ be iid $N(\mu, \sigma^2)$ random variables, where μ is known. Though it is hardly conceivable to know the exact value of μ in practice, let's consider median-unbiased estimation of σ^2 given the value of μ. Let $S^2 = \sum_{i=1}^{n} (X_i - \mu)^2$. Since $S^2/\sigma^2 \sim \chi^2_n$, $S^2/\chi^2_{n,.5}$ is median-unbiased for σ^2, where $\chi^2_{u,.5}$ denotes the median of the chi-square distribution with u degrees of freedom. Let $Q = S^2/\chi^2_{n,.5}$. The density function of Q is given by

$$g_Q(q) = \frac{(\chi^2_{n,.5})^{n/2}}{\sigma^2 2^{n/2}\Gamma(n/2)} \left(\frac{q}{\sigma^2}\right)^{\frac{n-2}{2}} \exp\left\{-\frac{\chi^2_{n,.5}}{2}\frac{q}{\sigma^2}\right\}.$$

Therefore the denominator of diffusivity is given by

$$2g_Q(\sigma^2) = \frac{2(\chi^2_{n,.5})^{n/2}}{\sigma^2 2^{n/2}\Gamma(n/2)} \exp\left\{-\frac{\chi^2_{n,.5}}{2}\right\},$$

of which the value can be obtained easily from the table of the chi-square distribution. The expectation of the partial derivative of the joint likelihood f with respct to σ^2 is given by

$$E\left|\frac{\partial \log f}{\partial \sigma^2}\right| = E\left|\frac{1}{2\sigma^2}\left\{\frac{\Sigma_i (x_i - \mu)^2}{\sigma^2} - n\right\}\right|$$

$$= \frac{1}{2\sigma^2} E|\chi^2_n - n|,$$

since $\Sigma_i (X_i - \mu)^2/\sigma^2 \sim \chi^2_n$. This expectation can be evaluated approximately by use of Monte Carlo integration technique. The following table shows a summary of the computation for some vales of n.

n	$\chi^2_{n,.5}$	$2\sigma^2 g_Q(\sigma^2)$	$\sigma^2 E\left\|\frac{\partial \log f}{\partial \sigma^2}\right\|$	ratio
1	.455	0.43	0.47	1.11
2	1.39	0.69	0.75	1.08
3	2.37	0.89	0.94	1.05
4	3.36	1.05	1.09	1.04
5	4.35	1.19	1.22	1.03

The median-unbiased estimator $S^2/\chi^2_{n,.5}$ does not attain the lower bound. It can be easily shown that is attains the bound asymptotically. When $n = 10$, the ratio turns out to be 1.002 approximately. □

3. OPTIMALITY CONDITIONS FOR LOCATION AND SCALE PARAMETERS

Assume that the regularity conditions (12) are satisfied by the family of distribution functions $\{F_\theta; \theta \in \Theta\}$, $\Theta \subset R$. We say that a median-unbiased estimator δ for $\tau(\theta)$ is *optimal* for the family $\{F_\theta; \theta \in \Theta\}$ if δ attains the lower bound.

The equality in Theorem 2 holds if and only if

$$\left| \int_{[x:\delta(x) \leqslant \tau(\theta)]} [(\partial/\partial\theta) \log f(x;\theta)] f(x;\theta)\, \mu(dx) \right|$$
$$= \int_{[x:\delta(x) \leqslant \tau(\theta)]} |(\partial/\partial\theta) \log f(x;\theta)| f(x;\theta) \mu(dx) \tag{14}$$

and

$$\left| \int_{[x:\delta(x) > \tau(\theta)]} [(\partial/\partial\theta) \log f(x;\theta)] f(x;\theta)\, \mu(dx) \right|$$
$$= \int_{[x:\delta(x) > \tau(\theta)]} |(\partial/\partial\theta) \log f(x;\theta)| f(x;\theta) \mu(dx) \tag{15}$$

or equivalently, if and only if the function $\log f(x;\theta)$ is monotone in θ on $[x:\delta(x) \leqslant \tau(\theta)]$, and the same holds as well on $[x:\delta(x) > \tau(\theta)]$.

In general, finding such a family of distributions which satisfies the condition (14) and (15) seems to be not trivial. We now show a condition under which we can find an optimal median-unbiased estimator of a location parameter.

Theorem 3. Let $X = (X_1, ..., X_n)$ be a sample of n iid random variables from a density of the form $f(x_1;\theta) = c \exp h(x_1 - \theta)$, where c is a constant, and h is strictly concave. Assume that the regularity conditions (12) are satisfied. If we take a median-unbiased estimator $\delta(X)$ of θ such that $\Sigma_{i=1}^{n} h'(X_i - \delta(X)) = 0$, then such a median-unbiased estimator δ attains the lower bound and is the maximum-likelihood estimator of θ. Conversely, if a median-unbiased estimator δ of θ attains the lower bound, then $\Sigma_{i=1}^{n} h'(X_i - \delta(X)) = 0$ holds. Moreover, if ρ is a strictly monotone transformation, the $\rho(\delta)$ is also an optimal median-unbiased estimator of $\rho(\theta)$.

Proof: Strict concavity of h implies that $(\partial/\partial\theta)\log f(X;\theta) = -\Sigma_{i=1}^{n} h'(X_i - \theta)$ is strictly decreasing in θ. Since $\delta(X)$ satisfies $\Sigma_{i=1}^{n} h'(X_i - \delta(X)) = 0$, then $\delta(X) \geqslant \theta$ for $[x:(\partial/\partial\theta)\log f(x;\theta) \geqslant 0]$, and $\delta(X) \leqslant \theta$ for $[x:(\partial/\partial\theta)\log f(x;\theta) \leqslant 0]$; i.e., δ satisfies (14) and (15). Obviously, δ is the maximum-likelihood estimator of θ. Conversely, if δ satisfies (14) and (15), then $\Sigma_{i=1}^{n} h'(X_i - \delta(X))$ is zero since $(\partial/\partial\theta)\log f(X;\theta)$ is strictly decreasing in θ. The last part follows from the fact that when δ is median-unbiased for θ, $\rho(\delta)$ is also median-unbiased for $\rho(\theta)$ for a strictly monotone transformation ρ. □

As was mentioned in Section 1, Stangenhaus and David [1978b] showed the sufficient condition in the first part of Theorem 3 with an additional assumption of symmetric h and verified that δ is the maximum-likelihood estimator of θ.

Example 4. For $N(\mu, 1), h(X_1 - \mu) = -(X_1 - \mu)^2/2$, which is strictly concave. $\Sigma\, h'(X_i - \mu) = 0$ if and only if $\Sigma\,(X_i - \mu) = 0$. Hence if we take $\delta(X) = \bar{X}$, then $\bar{X}$ is a median-unbiased estimator of μ and achieves the lower bound as was shown in Example 2. Note that $\bar{X} + d$, where d is a constant, is median-unbiased for $\theta + d$ and attains its lower bound since $\bar{X}$ and $\bar{X} + d$ are one-to-one. □

Example 5. Let $X_1, X_2, ..., X_n$ be a sample of n independent observations from a double exponential distribution: $f(x_1; \theta) = (1/2) \exp (-|x_1 - \theta|)$. Assume $n = 2k + 1$, where k is a positive integer. $-|x_1 - \theta|$ is strictly concave. $(\partial/\partial\theta) \log f(x; \theta) = (\#\text{ of } x_i's \geqslant \theta) - (\#\text{ of } x_i's \leqslant \theta) \equiv d$. If we take the sample median M as a median-unbiased estimator of θ, which makes d to be 0, then M achieves the lower bound. □

Example 6. Let X be a random variable with a density of the form $f(x; \theta) = (1/2) \exp h(x - \theta)$, with h defined by

$$h(x - \theta) = \begin{cases} -(x - \theta)^2 \pi/4 & \text{if } x \geqslant \theta \\ -|x - \theta| & \text{if } x < \theta. \end{cases}$$

Since θ is the median of f, X itself can be taken as a median-unbiased estimator of θ. Though f is not symmetric, X attains the bound. □

In Theorem 3, we have shown that there exists an optimal median-unbiased estimator of the location parameter, assuming that the scale parameter is known. We now consider a certain one-to-one transformation of such an optimal median-unbiased estimator to determine an optimal median-unbiased estimator of the scale parameter. Note that a scale problem can be converted to a location problem.

Theorem 4. Let $Z = (Z_1, Z_2, ..., Z_n)$ be a random sample of size n from a scale density of the form

$$f(z_1; \sigma) = \frac{1}{\sigma} k\left(\frac{z_1}{\sigma}\right), \quad z_1 > 0.$$

Let $X_i = \log Z_i$, $i = 1, ..., n$. If $X = (X_1, ..., X_n)$ satisfies the assumptions of Theorem 3, then $\exp\{\delta(\log Z_1, ..., \log Z_n)\}$ is an optimal median-unbiases estimator of σ and is also the maximum-likelihood estimator of σ.

Proof: By assumption, $\delta(X_1, ..., X_n)$ is an optimal median-unbiased estimator of $\log\sigma$. Since the exponential function is a one-to-one transformation and median-unbiasedness is invariant under one-to-one transformations, $\exp\{\delta(\log Z_1, ..., \log Z_n)\}$ is also an optimal median-unbiased estimator of σ. It is the maximum-likelihood estimtor of σ by the invariance property of maximum-likelihood estimation since $\delta(\log Z_1, ..., \log Z_n)$ is the maximum-likelihood estimator of $\log\sigma$ by Theorem 3. □

Example 7. Let $Z = (Z_1, Z_2, ..., Z_n)$ be a random sample of size n from a continuous density

$$f(z_1; \sigma) = \frac{1}{2(z_1 - \theta)} \exp\{-|\log((z_1 - \theta)/\sigma)|\}, \quad z_1 > \theta,$$

where θ is known. Assume that $n = 2k + 1$, $k = 0, 1, ...$. Let M be the sample median of $Z_1, Z_2, ..., Z_n$. Then M is an optimal median-unbiased estimator of σ. □

Example 8. Let $Z = (Z_1, Z_2, ..., Z_n)$ be a random sample of size n from a continuous density

$$f(z_1; \sigma) = \frac{1}{\sqrt{2\pi}(z_1 - \theta)} \exp\{-[\log((z_1 - \theta)/\sigma)]^2/2\}, \quad z_1 > \theta,$$

where θ is known. Then $(\Pi_{i=1}^n Z_i)^{1/n}$ is an optimal median-unbiased estimator of σ. □

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