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ON RANDOMIZED STOPPING TIMES

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ABSTRACT

In this note we give a proof of the fact that the extremal elements of the set of randomized stopping times are exactly the stopping times.

Key words: randomized stopping times; stopping times; extremal elements.

Classification AMS: 60G40, 60G57.

RESUMEN

En esta nota damos una demostración del hecho de que los elementos extremales del conjunto de los tiempos de paro aleatorizados son los tiempos de paro.

Palabras clave: tiempos de paro aleatorizados; tiempos de paro; elementos extremales.

Clasificación AMS: 60G40, 60G57.

1. INTRODUCTION

We know (cf. Edgar, Millet, Sucheston (1981); Ghoussoub (1982)) that the extremal elements of the set of randomized stopping times are the stopping times and this is relevant in the context of the optimal

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stopping problem (see Dalang (1984)). The aim of this note is to give an alternative proof of this fact. To this end, we give a new characterization of the extremal elements of the set of randomized stopping times, by using the notion of optimal projection for processes with indexes in $\mathbf{R}_+ \cup \{\infty\}$ (see section 2).

2. NOTATIONS. OPTIONAL PROJECTION

The following notation will be used throughout the note. $\mathbf{K} = \mathbf{R}_+ \cup \{\infty\}$ is the one-point compactification of the set of nonnegative real numbers and $\mathbf{B}$ is the σ-algebra of Borel subsets of $\mathbf{K}$.

Let $(\Omega, \mathbf{A}, \mathbf{P})$ be a complete probability space and let $\mathbf{F} = \{\mathbf{F}_z, z \in \mathbf{K}\}$ be an increasing right continuous family of sub-σ-algebras of $\mathbf{A}$, such that $\mathbf{F}_\infty = \mathrm{V}_z \mathbf{F}_z$ and $\mathbf{F}_0$ containst the null sets.

A stopping time τ is a map from Ω to $\mathbf{K}$ such that, for every t, the set $\{\tau \leqslant t\}$ belongs to $\mathbf{F}_t$.

The set of randomized stopping times was introduced by Baxter, Chacon (1977). A randomized random time μ is a probability measure on $\Omega \times \mathbf{K}$, such that its projection on Ω is $\mathbf{P}$ (see Baxter and Chacon (1977); Edgar, Millet and Sucheston (1981)).

To each randomized random time μ there is associated (see Ghoussoub (1982)) a non decreasing, null on the origin, right-continuous process A, such taht $A_\infty \equiv 1$, i.e. $d\mu = dP \times A(\omega, dz)$. If this process is adapted, we say that μ is a randomized stopping time.

For every $\mu \in \Gamma$, Γ is the set of randomized stopping times, and for all measurable processes X, we shall write

$$\langle X, \mu \rangle = \int_{\Omega \times \mathbf{K}} X \, d\mu = E\left(\int_{\mathbf{K}} X_t \, dA_t \right)$$

where A is the process associated to μ.

In order to give a characterization of the extremal elements of Γ we need the notion of optimal projection for process with indices in $\mathbf{K}$.

Let O the optional σ-algebra on $\Omega \times \mathbf{R}_+$ and let P the optional projection operator on $\Omega \times \mathbf{R}_+$ (for the definition see Dellacherie, Meyer (1980)). We extend these notions on $\Omega \times \mathbf{K}$, by the following.

Definition 1

The σ-algebra $\hat{O} = O \vee \sigma\{A \times \{\infty\}, A \in F_\infty\}$ is the optional σ-algebra on $\Omega \times \mathbf{K}$; and the operator $\hat{P}$ defined by $(\hat{P}(X))_t = (P(X))_t$ if $t \in \mathbf{R}_+$, and by $(\hat{P}(X))_\infty = E(X_\infty \mid F_\infty)$, for each measurable bounded process X, is the optional projection operator.

Moreover, a random measure μ on $\Omega \times \mathbf{K}$ is optional if and only if its nondecreasing associated process is optional on $\Omega \times \mathbf{K}$.

We can prove, as in the $\Omega \times \mathbf{R}_+$ case, that if X is a bounded process, $\hat{P}(X)$ is the only bounded, optional process Y such that, for all stopping times τ, $E(Y_\tau) = E(X_\tau)$. Moreover, the σ-algebra $\hat{O}$ is generated by the adapted, right-continuous processes, and a random measure μ on $\Omega \times \mathbf{K}$ is optional if and only if, for all bounded processes X,

$$\langle X, \mu \rangle = \langle \hat{P}(X), \mu \rangle$$

3. EXTREME ELEMENTS OF THE SET OF RANDOMIZED STOPPING TIMES

Theorem 2

Let μ an element of Γ and let A be its associated nondecreasing process. Then, the following conditions are equivalent

(i) μ is an extremal element of Γ.

(ii) If g is an optional bounded function from $\Omega \times \mathbf{K}$ to $\mathbf{R}$, such that for every $F \in \mathbf{A}$, $\int_{F \times K} g \, d\mu = 0$, then $g = 0$ $\mu - a.s.$

(iii) There exists a stopping time τ such that

$$\mathrm{A}(\omega, [0, t]) = I_{[\tau, \infty]}(t), \text{ for all } (\omega, t) \in \Omega \times \mathbf{K}$$

Proof

It is obvious that (iii) implies (i). To check (i) implies (ii) let g be an optional function from $\Omega \times \mathbf{K}$ to $\mathbf{R}$, such that $|g| \leqslant k$ and for all $F \in \mathbf{A}$, $\int_{F \times \mathbf{K}} g \, d\mu = 0.$

Define,

$$d\mu_1 = \left(1 + \frac{g}{2k}\right)d\mu \quad \text{and} \quad d\mu_2 = \left(1 - \frac{g}{2k}\right)d\mu$$

This measures verifie, for all

$$F \in \mathbf{A},\ \langle F, \mu_i \rangle = \langle F, \mu \rangle + \langle F \frac{g}{2k}, \mu \rangle = \langle F, \mu \rangle = \mathbf{P}(F)$$

and if X is a bounded process,

$$\langle X, \mu_i \rangle = \langle X, \mu \rangle + \langle X \frac{g}{2k}, \mu \rangle$$

bearing in mind that μ and g are optional

$$\langle X, \mu_i \rangle = \langle \hat{P}(X), \mu \rangle + \langle \hat{P}(X) \frac{g}{2k}, \mu \rangle = \langle \hat{P}(X), \mu_i \rangle$$

so this measures belong to Γ and $\mu = \frac{1}{2}\mu_1 + \frac{1}{2}\mu_2$.

To prove (ii) implies (iii), define for all $(\omega, t) \in \Omega \times \mathbf{K}$,

$$g(\omega, t) = A(\omega, [0, t]) + A(\omega, [0, t)) - 1$$

Then g is optional, bounded and satisfies for all $F \in \mathbf{A}$, $\int_{F \times \mathbf{K}} g\, d\mu = 0$ (see Dellacherie, Meyer (1980), pp. 6/90). Hence, $g = 0$ $\mu - a.s.$

This implies that, ω – a.s.

$$A(\omega, [0, t]) + A(\omega, [0, t)) - 1 = 0,\ A(\omega, \cdot) - a.s. \tag{1}$$

As the process A is increasing, it holds $\omega - a.s.$ that there exists a point t_ω such that $A(\omega, t_\omega) = 1$. So $\omega - a.s.$ $A(\omega, dt)$ is a Dirac measure. Then $A_t = I_{[\tau, \infty]}(t)$ where $\tau(\omega) = t_\omega$ and as A is adapted, τ is a stopping time. ■

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