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NEGATIVE DEPENDENCE STRUCTURES THROUGH STOCHASTIC ORDERING

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ABSTRACT

Several new multivariate negative dependence concepts such as negatively upper orthant dependent in sequence, negatively associated in sequence, right tail negatively decreasing in sequence, and upper (lower) negatively decreasing in sequence through stochastic ordering are introduced. These concepts conform with the basic idea that if a set of random variables is split into two sets, then one is «increasing» whenever the other is «decreasing». Our concepts are easily verifiable and enjoy many closure properties. Applications to probability and statistics are also considered.

1. INTRODUCCION AND SUMMARY

The concept of negatively dependent in sequence through stochastic ordering (NDS) random variables was introduced into the statistical literature by Block, Savits and Shaked (1985) (see also, Joag-Dev and Proschan, 1983 and references there).

In this paper we introduce several new multivariate concepts (see Section 2 for exact definitions). The main motivation for our definitions is to follow the intutive requierement that if a set of negatively dependent random variables is split into two subsets in some manner then one subset will tend to be «large» when the other subset is «small» and vice versa. In Section 2, we introduce several new types of negative dependence, and develop their properties. As will be seen in Sections 3 and 4, our conditions are often easily verifiable, they arise naturally in many

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applications and enjoy some closure properties which enable us to derive useful inequalities for many well known distributions.

In the sequel the following two well known results, which are useful in their own right, will be used.

Let (X, Y) be a pair of real random variables and Z be a real or vector valued random variable. Then

$$\mathrm{Cov}\,(X, Y) = E\{\mathrm{Cov}\,(X, Y \mid Z)\} + \mathrm{Cov}\,\{E(X \mid Z), E(Y \mid Z)\}. \tag{1.1}$$

Let X be a real random variable. For every pair of increasing functions f, g:

$$\mathrm{Cov}\,[f(X), g(X)] \geqslant 0. \tag{1.2}$$

For f and g discordant functions, the inequality is reversed. (Two functions are discordant if one is increasing and other is decreasing). Inequality (1.2) is well known as Techebycheff's inequality.

Throughout this paper, we use «increasing» in place of «nondecreasing» and «decreasing» in place of «nonincreasting». Vectors in R^n are denoted by $x = (x_1, ..., x_n)$ and $x \leqslant y$ means $x_i \leqslant y_i$, $i = 1, ..., n$. Similarly x^i denotes $(x_1, ..., x_{i-1}, x_{i+1}, ..., x_n)$ and $x > y$, means $x_i > y_i$, $i = 1, ..., n$. A real function on R^n will be called increasing if it is increasing in each variable when the other variables are held fixed.

2. NEGATIVE DEPENDENCE CONCEPTS

Definition 2.1. (Ebrahimi and Ghosh, 1981). The random variables $X_1, ..., X_n$ (or the random vector X or its distribution function) are said to be negatively upper orthant dependent (NUOD) if for every x,

$$P[X > x] \leqslant \prod_{i=1}^{n} P(X_i > x_i) \tag{2.1a}$$

They are said to be negatively lower orthant dependent (NLOD) if for every x,

$$P[X \leqslant x] \leqslant \prod_{i=1}^{n} P(X_i \leqslant x_i) \tag{2.1b}$$

When $n = 2$, (2.1a) and (2.1b) are equivalent, but not when $n \geqslant 3$ (see, e.g., Ebrahimi and Ghosh, 1981). If both (2.1a) and (2.1b) are satisfied, then $X_1, ..., X_n$ are said to be negatively orthant dependent (NOD).

Definition 2.2. The random variables $X_1, ..., X_n$ are said to be negatively upper orthant dependent in sequence (NUODS) if for every $i = 1, 2, ..., n$ and every x,

$$P(X > x) \leqslant P(X^i \leqslant x^i)P(X_i > x_i).$$

They are said to be negatively lower orthant dependent in sequence (NLDOS) if for every $i = 1, ..., n$ and every x,

$$P(X \leqslant x) \leqslant P(X^i \leqslant x^i)P(X_i \leqslant x_i). \tag{2.2b}$$

If both (2.2a) and (2.2b) are satisfied, then we say that $X_1. ..., X_n$ are negatively orthant dependent in sequence (NODS).

Definition 2.3. The random variables $X_1, ..., X_n$ are said to be negatively associated in sequence (NAS) if

$$\text{Cov}\,[f(X^i),\ g(X_i)] \leqslant 0 \tag{2.3}$$

for every pair of increasing functions f, g defined on R^{n-1}, R respectively, and every $i = 1, ..., n$.

Definition 2.4 (Joao-Dev and Proschan, 1983). The random variables $X_1, ..., X_n$ are said to be negatively associated (NA) if for every pair of disjoint subsets A_1, A_2 of $\{1, 2, ..., n\}$

$$\text{Cov}\,\{f_i(X_i,\ i \in A_1),\ f_2(X_j,\ j \in A_2)\} \leqslant 0 \tag{2.4}$$

whenever f_1 and f_2 increasing.

Definition 2.5. The random variables $X_1, ..., X_n$ are said to be right tail negatively decreasing in sequence (RTNDS) if

$$P[X^i > x^i \mid X_i > x_i] \tag{2.5}$$

is decreasing in x_i for all real x^i, and every $i = 1, ..., n$.

Definition 2.6. The random variables $X_1, ..., X_n$ are said to be left tail negatively increasing in sequence (LTNIS) if

$$P[X^i \leqslant x^i \mid X_i \leqslant x_i] \tag{2.6}$$

is increasing in x_i for all real x^i, and every $i = 1, ..., n$.

Definition 2.7 (Block, Savits and Shaked, 1985). The random variables $X_1, ..., X_n$ are said to be negatively dependent through stochastic ordering (NDS) if

$$E[f(X^i) \mid X_i = x_i] \tag{2.7}$$

is decreasing in x_i *for all increasing functions* f defined on R^{n-1} and every $i = 1, ..., n$.

Definition 2.8. The random variables $X_1, ..., X_n$ are said to be upper negatively decreasing in sequence through stochastic ordering (UNDSS) if

$$P[X^i > x^i \mid X_i = x_i] \tag{2.8}$$

is decreasing in x_i for all real vectors x^i, and every $i = 1, ..., n$.

Example (The Multinomial Distribution). Suppose that

$$P[X_i = x_i,\ i = 1, ..., n]$$
$$= \binom{N}{x_0, ..., x_n} \prod_{i=0}^{n} p_i^{x_i},$$

where

$$x_0 = N - \sum_{i=1}^{n} x_i \quad \text{and} \quad p_0 = 1 - \sum_{i=1}^{n} p_i.$$

Without loss of generality, it is easy to verify that

$$P[X_i > c_i,\ i = 1, ..., n-1 \mid X_n = t]$$

is decreasing in t, so that $X_1, ..., X_n$ are UNDSS.

Definition 2.9. The random variables $X_1, ..., X_n$ are said to be lower negatively increasing in sequence through stochastic ordering (LNISS) if

$$P[X^i \leqslant x^i \mid X_i = x_i] \tag{2.9}$$

is increasing in x_i for all real vectors x^i, and every $i = 1, \ldots, n$.

Example (Ranks). Let X_i, $i = 1, \ldots, n$, be a random sample from a continuous distribution and R_i, $i = 1, \ldots, n$, be their ranks. Let

$$c_1 \leqslant c_2 \leqslant \cdots \leqslant c_m \leqslant n$$

be m positive integers. It can be shown that

$$P[R_i \leqslant c_i,\ i = 1, \ldots, m] = \frac{c_1}{n} \cdot \frac{c_2 - 1}{n - 1} \cdots \frac{c_m - m + 1}{n - m + 1}.$$

Using this it can be shown that

$$P[R_i \leqslant c_i,\ i = 1, \ldots, m \mid R_{m+1} = t]$$

is increasing in t, so that $R_1, \ldots, R_n$ are LNISS.

Remark 2.10. Since any of the new introduced concepts imply that

$$P(X_i > x_i,\ X_j > x_j) \leqslant P(X_i > x_i) P(X_j > x_j) \quad \text{for} \quad i \leqslant i \leqslant j \leqslant n, \tag{2.11}$$

it follows, as in Lehmann (1966), that $\operatorname{Cov}(X_i, X_j) \leqslant 0$. This justifies the name negative dependence for any of the introduced concepts.

When the monotonicity direction in (2.2a), (2.2b), (2.3), (2.5), (2.6) and (2.9) are reversed, analogs of positive dependence are obtained.

3. RELATIONS

We will now derive some relations among the introduced concepts.

Theorem 3.1.

(a) NDS implies both UNDSS and LNISS,

(b) UNDSS $\Rightarrow$ RTNDS $\Rightarrow$ NUODS $\Rightarrow$ NUOD, and

(c) LTNDS $\Rightarrow$ NLODS $\Rightarrow$ NUOD.

Proff.

(a) *Obvious.*

(*b*) *The implications RTNDS ⇒ NUODS ⇒ NUOD are obvious from the definitions. To see that UNDS ⇒ RTNDS, let*

$$h(x_i) = P[X^i > x^i \mid X_i = x_i]$$

and

$$g(x_i) = P[X^i > x^i \mid X_i > x_i].$$

Observe that

$$g(x_i) = \int_{x_i}^{\infty} h(t)\, dF_{X_i}(t) \Big/ \int_{x_i}^{\infty} dF_{X_i}(s).$$

Also, $g(x_i) \leqslant h(x_i)$, since h is decreasing by assumption and g is a weighted average of h.

Next assume $y_i \leqslant x_i$ and write

$$g(y_i) = \int_{x_i}^{\infty} h(t)\, dF_{X_i}(t) \Big/ \int_{y_i}^{\infty} dF_{X_i}(s) + \int_{y_i}^{x_i} h(t)\, dF_{X_i}(t) \Big/ \int_{y_i}^{\infty} dF_{X_i}(s) \geqslant$$

$$\geqslant g(x_i) \int_{x_i}^{\infty} dF_{X_i}(t) \Big/ \int_{y_i}^{\infty} dF_{X_i}(s) + h(x_i) \int_{y_i}^{x_j} dF_{X_i}(t) \Big/ \int_{y_i}^{\infty} dF(s) =$$
$$= g(x_j).$$

Thus g, is decreasing, and the proof is completed.

(c) The proof is similar to that of (b), with obvious modifications.

Remark 3.2. It is not hard to show that NDS ⇒ LNISS.

Theorem 3.3. NDS ⇒ NAS.

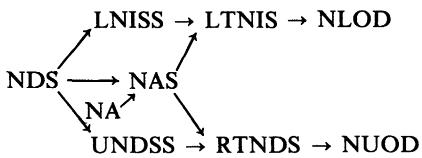
Proof. Consider the identity (1.1), and write

$$\operatorname{Cov}[f(X^i), g(X_i)] = \operatorname{Cov}\{E[f(X^i) \mid X_i] \cdot E[g(X_i) \mid X_i]\} + \\ + E\{\operatorname{Cov}[f(X^i), g(X_i) \mid X_i]\}. \qquad (3.1)$$

Suppose now that (2.6) holds. Then the expected values in the first term on the right side of (3.1) are discordant functions in X_i (almost surely) so that in view of (1.2), this term is nonpositive. Further, the conditional covariance in the second term is zero, and the same holds for its expected value.

Remark 3.4. It is obvious that NA implies NAS.

We may now summarize the above implications in the following chains:



In some applications negative dependence is created when the random variables are subjected to conditioning, as in the following theorem.

Theorem 3.5. Let $X_1, ..., X_n$ be independent and suppose the conditional expectation $E\{f(X_i, i \in A) \mid \sum_{i \in A} X_i\}$ is increasing in $\sum_{i \in A} X_i$ for every increasing function f and every subset A of order one or $n - 1$ of $\{1, ..., n\}$. Then the conditional distribution of $X_1, ..., X_n$ given $\sum X_i$, is NAS almost surely.

Proof. Let A be an arbitrary subset of order $n - 1$ of $\{1, ..., n\}$. Let $S_1 = \sum_{i \in A} X_i, S_2 = X_j, \{j\} = \bar{A}, S = S_1 + S_2$, and f_1, f_2 be a pair of increasing functions. Using (1.1), where the conditioning vector is taken as (S_1, S_2), it follows that

$$\operatorname{Cov}\{f_1(X_i, i \in A), f_2(X_j) \mid S\} =$$
$$\operatorname{Cov}\{E(f_1 \mid S_1, S_2), E(f_2 \mid S_1, S_2) \mid S\}.$$

With $S = S_1 + S_2$, the two terms inside the brackets on the right side are discordant functions of S_1 and hence, by (1.2), the covariance is negative.

4. PRESERVATION RESULTS AND APPLICATIONS

Closure theorems are useful for identifying negatively dependent distributions or for constructing new negatively dependent distributions from known ones. In this section we introduce some closure results and some illustrative applications.

The following definitions are needed for the statements of Result 4.1 and Result 4.2.

A bivariate function $k(...)$ which is defined on $S_1 \times S_2$ (where S_1, S_2 are subsets of R) is said to be totally positive of order $2(TP_2)$ on $S_1 \times S_2$ if $k(x, y) \geqslant 0$ and if

$$k(x_1, y_1)k(x_2, y_2) \geqslant k(x_1, y_2)k(x_2, y_1),$$

whenever $x_1 \leqslant x_2$ and $y_1 \leqslant y_2$.

The function k is said to be reverse regular of order 2 (RR_2) on $S_1 \times S_2$ if $k(x, y) \geqslant 0$ and if

$$k(x_1, y_1)k(x_2, y_2) \leqslant k(x_1, y_2)k(x_2, y_1)$$

whenever $x_1 \leqslant x_2$, $y_1 \leqslant y_2$ (see Karlin, 1968, p. 12).

A univariate density f is said to be a Polya frequency of order 2 (PF_2) if ($f(x - y)$ is TP_2 on $R \times R$. A probability mass function is PF_2 if $f(x - y)$ is TP_2 on $N \times N$ where $N = \{..., -1, 0, 1, ...\}$. A thorough discussion of PF_2 density functions and many examples can be found in Karlin (1968). Next we present two results which give rise to our introduced concepts of negative dependence. The Dirichlet, multinomial, multivariate hypergeometric, multivariate negative binomial and various negatively correlated normals are example of distributions that can be generated via such results.

The following two results are direct applications of Model 1 of Block, Savits, and Shaked (1985).

Result 4.1. If $Y_1, ..., Y_{n+1}$ are indepdendent random variables with PF_2 densities (or probability functions) then the random vector $(X_1, ..., X_n)$ which admits the representation

$$(X_1, ..., X_n) \overset{\text{st}}{=} (Y_2, ..., Y_{n+1}) \mid a_1 Y_1 + \cdots + a_{n+1} Y_{n+1} = t$$

for some constants t and $a_i > 0$, $i = 1, ..., n + 1$ is (*), where (*) is one of the following: RR_2 in pairs, UNDDS, RTNDS, NAS, NODS, NOD. ($X \overset{\text{st}}{=} Y$ means that X and Y have the same distribution).

Result 4.2. Let $X_1, ..., X_n$ be i.i.d. random variables having a continuous distribution F. Let $X_{(i)}$ be the ith order satistic and

$$(Y_1, ..., Y_n) \stackrel{\text{st}}{=} (X_1, ..., X_n) \mid \{X_{(i)} = t_1, ..., X_{(i_n)} = t_n\}$$

for $i \leqslant i < \cdots < i_n < n$ and $t_1 < \cdots < t_n$. Then $(Y_1, ..., Y_n)$ is (*) where (*) is the same as in Result 4.1.

Block, Savits and Shaked (1985) have also shown that: If

$$(X_1, ..., X_n) \stackrel{\text{st}}{=} (Y_1 - \bar{Y}, ..., Y_n - \bar{Y})$$

where $Y_1, ..., Y_n$ are i.i.d. random variables and $\bar{Y} = n^{-1} \sum_{i=1}^{n} Y_i$, then $(X_1, ..., X_n)$ is NDS. It follows from Theorems 3.1 and 3.2 that $(X_1, ..., X_n)$ is also UNDSS, RTNDS, NAS, NUODS and NUOD. It should be noted that the multivariate normal distribution with negative correlations satisfies the above representation.

Theorem 4.3. Assume that $(X_1, ..., X_n)$ and $(Y_1, ..., Y_n)$ are independent UNDSS. If all the univariate marginal densities (with respect to Lebesgue measure), or probability functions in the discrete case, of X and Y are PF_2, then

$$(X_1 + Y_1, ..., X_n + Y_n) \text{ is UNDSS.}$$

The proof of Theorem 4.3 is easily obtained from the following lemmas.

Lemma 4.4. Let $X_1, ..., X_n$ be independent random variables with PF_2 densities or probability functions. Then

$$(X_1, ..., X_n) \uparrow \text{st. } X_1 + \cdots + X_n.$$

Proof. See Efron (1965).

Lemma 4.5. Let $X = (X_1, ..., X_n)$ and $Y = (Y_1, ..., Y_n)$ be independent and assume

$$P[X_1, ..., X_{n-1}) > (s_1, ..., s_{n-1}) \mid X_n = s_n] \text{ is} \downarrow \text{in } s_n \tag{4.1}$$

and

$$P[(Y_1, ..., Y_{n-1}) > (t_1, ..., t_{n-1}) \mid Y_{n-1} = t_n] \text{ is} \downarrow \text{in } t_n. \quad (4.2)$$

Further assume taht X_n and Y_n have PF_2 densities of probability functions. Then

$$P[(X_1 + Y_1, ..., X_{n-1} + Y_{n-1}) > (U_1, ..., U_{n-1}) \mid X_n + Y_n = U_n]$$

is decreasing in U_n for all $U_1, ..., U_{n-1}$.

Proof. Clearly

$$P[X_1 + Y_1 > U_1, ..., X_{n-1} + Y_{n-1} > U_{n-1} \mid X_n + Y_n = U_n]$$
$$E = [\phi(X_n, Y_n) \mid X_n + Y_n = U_n],$$

where

$$\phi(x_n, y_n) = E[X_1 + Y_1 > U_1, ..., X_{n-1} + Y_{n-1} > U_{n-1} \mid X_n = x_n, Y_n = y_n].$$

However, $\phi(x_n, y_n)$ is decreasing in x_n and in y_n because of (4.1), (4.2) and independence. Thus, by Lemma 4.5,

$$E[\phi(X_n, Y_n) \mid X_n + Y_n = U_n]$$

is decreasing in U_n.

A straight forward generalization of RTNDS can be made by requiring

$$P[X_i > x_i, i \in C \mid X_j > x_j, j \in \bar{C}] \quad (4.3)$$

be decreasing in x_j, $j \in C$ for every partition $\{C, \bar{C}\}$ of the index set $\{1, 2, ..., n\}$ and every x.

Random variables satisfying (4.3) will be called right negatively dependent in complements (RNDC).

The following theorem gives a sufficient condition, which is easy to check, for X to be RNDC and hence RTNDS.

Theorem 4.6. Let $\bar{F}(x_1, ..., x_n) = P[X_i > x_i, i = 1, 2, ..., n]$ be RR_2 in each pair of arguments for fixed values of the remaining arguments. Then X is RNDC. Moreover, every permutation of $X_1, ..., X_n$ is RTNDS.

Proof. Fix $i < r < n$. It is enough to show that

$$P[X_i > x_i, i = 1, ..., r \mid X_j > x_j, j = r + 1, ..., n]$$

is decreasing in $x_{r+1}, ..., x_n$ for all real numbers $x_1, ..., x_n$. Note that

$$P[X_i > x_i, i = 1, ..., r \mid X_j > x_j, j = r + 1, ..., n] = \frac{\bar{F}(x_1, ..., x_n)}{\bar{F}(-\infty, ..., -\infty, x_{r+1}, ..., x_n)}.$$

$$\frac{\bar{F}(x_1, ..., x_n)}{\bar{F}(-\infty, x_2, ..., x_n)} \cdot \frac{\bar{F}(-\infty, x_2, ..., x_n)}{\bar{F}(-\infty, -\infty, x_3, ..., x_n)} \cdot \frac{\bar{F}(-\infty, ..., x_r, ..., x_n)}{\bar{F}(-\infty, ..., -\infty, x_{r+1}, ..., x_n)}.$$

By the RR_2 property, each of the above multiplicants is desreasing in $x_{r+1}, ..., x_n$. Thus, the result follows.

Remark 4.7. It should be remarked that karlin (1968) has shown that if the joint denity (or probability function) of X exists and is RR_2 in pars, then so is $\bar{F}(x_1, ..., x_n)$.

An example of how Remark 4.7 might be useful is the following.

Let $X_{(1)} \leqslant \cdots \leqslant X_{(n)}$ be the order statistics of a sample of size n from a population with a PF_2 density. Let

$$\begin{aligned} Y_1 &= nX_{(1)}, \\ Y_2 &= (n-1)(X_{(2)} - X_{(1)}), \\ \vdots &= \\ Y_n &= X_{(n)} - X_{(n-1)} \end{aligned}$$

be the normalized spacings. It can be shown that the joint density of $Y_1, ..., Y_n$is RR_2 in pairs. Thus Y es RNDC.

The following two results are useful for constructing new negatively dependent distribution functions from known ones.

Proposition 4.8. (a) Independent random variables are (τ),

(b) if $X_1, ..., X_n$ are (τ), then

(i) any subset of two or more of $X_1, ..., X_n$ is (τ)

(ii) if $g_1, ..., g_n$ are strictly increasing functions, the $g_1(X_1), ..., g_n(X_n)$ are (τ), where (τ) is one of the following: NODS, NAS, RTNDS, or UNDSS.

Proposition 4.9. If $(X_1, ..., X_n)$ and $(Y_1, ..., Y_n)$ are independent and are (τ) then $(X_1, ..., X_n, Y_1, ..., Y_n)$ is (τ) where (τ) is the same as in proposition 4.8.

The proofs of these propositions are straightforward and will be omitted.

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