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BAYES ESTIMATION OF THE RELIABILITY FUNCTION AND HAZARD RATE OF A WEIBULL FAILURE TIME DISTRIBUTION

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SUMMARY

Given the recorded life times from a Weibull distribution , Bayes estimates of the reliability function and hazard rate are obtained using the posterior distributions and some recent results on Bayesian approximations due to Lindley (1980). Based on a Monte Carlo study, these estimates are compared with their maximum likelihood counterparts.

Key words: Jeffreys' prior; Posterior expectations; Simulation; Squared-error deviation.

A.M.S. Subject Classification: Primary 62F15; Secondary 62N05.

Título: Estimación bayesiana de la función de fiabilidad y de la tasa de riesgo de una distribución de Weibull de tiempo de fallos.

RESUMEN

Utilizando las distribuciones a posteriori y ciertos resultados recientes sobre aproximaciones bayesianas debidas a Lindley, se obtienen estimadores Bayes de la función de fiabilidad, dados los tiempos de vida. Basado en un estudio mediante técnicas de Monte Carlo, estos estimadores se comparan con los máximo verosímiles.

Palabras clave: A priori de Jeffreys; esperanza a posteriori; simulación; desviación cuadrática media.

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1. INTRODUCTION

Consider the Weibull probability density function (pdf)

$$f(x|\theta, p) = \frac{p}{\theta} x^{p-1} \exp\left(-\frac{x^p}{\theta}\right), \quad x,\ p,\ \theta > 0 \tag{1.1}$$

Weibull distribution has been extensively used in life testing and reliability studies on strength of material (Weibull, 1939, 1951). Cohen (1965), Mann, Schafer and Singpurwalla (1974) and Sinha (1986) discuss a variety of situations in which this distribution has been used for the analysis of other types of failure data.

By definition, the reliability function and the hazard rate at any time t are respectively given by

$$R_t = \exp\left(-\frac{t^p}{\theta}\right) \quad \text{and} \quad \mu_t = \frac{p}{\theta} t^{p-1}$$

With the shape parameter $p > 1$, μ_t is an increasing function of time and this property makes the Weibull distribution especially suitable for life testing experiments of componnents which age with time. Cohen (1965) obtained an iterative solution $(\hat{\theta}, \hat{p})$ of the likelihood equations. Having obtained $(\hat{\theta}, \hat{p})$, it is easy to compute the maximum likelihood estimate (MLE) $\hat{R}_t$. One might like to compare $\hat{R}_t$ with the corresponding Bayesian estimator R_t^*. Under squared-error-loss, R_t^* is the posterior expectation of R_t and is given by the ratio of two integrals which cannot be expressed in a simple form. Lindley (1980) developed asymptotic expansions of the ratio of two integrals whereby one can estimate a function of the parameters of interest.

The object of this paper is to compare the MLE of R_t and μ_t with their Bayesian counterparts using the posterior distributions and those obtained from Bayesian approximations (Lindley, 1980).

2. POSTERIOR DISTRIBUTION

Justified on the grounds of its invariance under parametric transformations, we use Jeffreys' (1981) prior for (θ, p) as given by

$$g(\theta, p) \propto \sqrt{|I(\theta, p)|}$$

where

$$|I(\theta, p)| = -E \begin{vmatrix} \partial^2 \log f(x|\theta, p)/\partial\theta^2 & \partial^2 \log f(x|\theta, p)/\partial\theta\,\partial p \\ \partial^2 \log f(x|\theta, p)/\partial\theta\,\partial p & \partial^2 \log f(x|\theta, p)\partial p^2 \end{vmatrix} =$$

$$= \begin{vmatrix} 1/\theta^2 & \{\psi(2) + \log\theta\}/\theta p \\ \{\psi(2) + \log\theta\}/\theta p & \{1 + \psi^2(2) + c + (\log\theta)^2 \\ & + 2\psi(2)\log\theta\}/p^2 \end{vmatrix}$$

$$= (1 + c)/\theta^2 p^2$$

where

$$\psi(2) = \int_0^\infty u \exp(-u)(\log u)\, du \quad \text{and} \quad c = 0.6449.$$

Thus $g(\theta, p) \propto 1/\theta p$.

Given a sample $\underline{x} = (x_1, x_2, ..., x_n)$ from the pdf (1.1), the likelihood function

$$\ell(x|p, \theta) = p^n \lambda^{p-1} \exp\left\{-\sum_{i=1}^n x_i^p/\theta\right\}/\theta^n, \; \lambda = \sum_{i=1}^n x_i.$$

The joint posterior of (p, θ) is given by

$$\pi(\theta, p|x) = K\ell(x|p, \theta)g(\theta, p)$$

where K is a normalizing constant. Under a squared-error loss function,

$$R_t^* = E(R_t|\underline{x})$$

$$= \frac{\int_0^\infty \int_0^\infty \exp\left(-\frac{t^p}{\theta}\right) \ell x|p, \theta) g(\theta, p)\, d\theta\, dp}{\int_0^\infty \int_0^\infty \ell(x|p, \theta) g(\theta, p)\, d\theta\, dp} \qquad (2.1)$$

$$= \frac{\displaystyle\int_0^\infty \frac{p^{n-1}\lambda^{p-1}}{\left(\sum_{i=1}^{n} x_i^p + t^p\right)^n}\,dp}{\displaystyle\int_0^\infty \frac{p^{n-1}\lambda^{p-1}}{\left(\sum_{i=1}^{n} x_i^p\right)^n}\,dp} \tag{2.2}$$

Similarly,

$$\mu_t^* = E(\mu_t|x)$$

$$= \frac{n\displaystyle\int_0^\infty \frac{p^n(\lambda t)^{p-1}}{\left(\sum_{i=1}^{n} x_i^p\right)^n}\,dp}{\displaystyle\int_0^\infty \frac{p^{n-1}\lambda^{p-1}}{\left(\sum_{i=1}^{n} x_i^p\right)^n}\,dp}. \tag{2.3}$$

(2.2) and (2.3 cannot be expressed in simple closed forms. In as much as the estimates R_t^* and μ_t^* have to be numerically approximated using innovative programming and expensive computer time, we might as well consider another approximations to the radio of two integrals such as (2.1) due to Lindley (1980) which is easier to compute on an automatic desk calculator.

3. LINDLEY'S EXPANSION

Lindley (1980) considered an approximation for the ratio of integrals of the form

$$\int w(\theta)\exp\{L(\theta)\}\,d\theta / \int v(\theta)\exp\{L(\theta)\}\,d\theta \tag{3.1}$$

where $\theta = (\theta_1, \theta_2, ..., \theta_n)$ is the parameter, $L(\theta)$ is the logarithm of the likelihood function and $w(\theta)$ and $v(\theta)$ are arbitrary.

Let $v(\theta)$ be the prior distribution of θ and $w(\theta) = u(\theta)v(\theta)$. From (3.1) we have the posterior expectation

$$E\{u(\theta)|\underline{x}\} = \int u(\theta) \exp\{L(\theta) + \rho(\theta)\}\, d\theta / \int \exp\{L(\theta) + \rho(\theta)\}\, d\theta \quad (3.2)$$

where $\rho(\theta) = \log\{v(\theta)\}$.

Note that (2.1) has the same form as (3.2) with θ replaced by

$$\underline{\theta} \equiv (\theta, p), \quad u(\underline{\theta}) = \exp\left(-\frac{t^p}{\theta}\right), \quad \rho(\underline{\theta}) = -\log p - \log\theta$$

Since θ and p are locally orthogonal, Lindley's asymptotic expansion of (3.2) leads to

$$R_t^{**} = E\{u(\theta, p)|\underline{x}\} = u + \frac{1}{2}(u_{11}\sigma_{11} + u_{22}\sigma_{22}) + u_1\rho_1\sigma_{11}$$

$$+ u_2\rho_2\sigma_{22} + \frac{1}{2}(L_{30}u_1\sigma_{11}^2 + L_{03}u_2\sigma_{22}^2)$$

where $u = \exp\left(-\dfrac{t^p}{\theta}\right)$.

$$u_1 = \frac{\partial u}{\partial\theta} = u\frac{t^p}{\theta^2}, \quad u_2 = \frac{\partial u}{\partial p} = -\frac{ut^p(\log t)}{\theta},$$

$$u_{11} = \frac{\partial^2 u}{\partial\theta^2} = \frac{ut^p}{\theta^3}\left(\frac{t^p}{\theta} - 2\right),$$

$$u_{22} = \frac{\partial^2 u}{\partial p^2} = \frac{ut^p(\log t)^2}{\theta}\left(\frac{t^p}{\theta} - 1\right), \quad \rho_1 = \frac{\partial\rho}{\partial\theta} = -\frac{1}{\theta},$$

$$\rho_2 = \frac{\partial\rho}{\partial p} = -\frac{1}{p}, \quad \sigma_{11} = (-L_{20})^{-1}, \quad \sigma_{22} = (-L_{02})^{-1},$$

$$L_{20} = \frac{\partial^2 L}{\partial\theta^2} = -\frac{n}{\theta^2}$$

$$L_{02} = \frac{\partial^2 L}{\partial p^2} = -\frac{n}{p^2} - \frac{\sum_{i=1}^{n} x_i^p(\log x_i)^2}{\theta},$$

$$L_{30} = \frac{\partial^3 L}{\partial\theta^3} = \frac{4n}{\theta^3}$$

$$L_{03} = \frac{\partial^3 L}{\partial p^3} = \frac{2n}{p^3} - \frac{\sum_{i=1}^{n} x_i^p (\log x_i)^3}{\theta}$$

all evaluated at the mle $(\hat{\theta}, \hat{p})$.

For numerical computations, Lindley uses the relationship $\partial^r u(\theta)/\partial\theta^r \approx \Delta^r u(\theta)/h^r$ where h, the interval of differences, is small.

$L(p, \theta)$ is evaluated for a range of values of p and θ in the neighbourhood of $(\hat{p}, \hat{\theta})$ and the differences are used to obtain $L_{ij}(\hat{p}, \hat{\theta})$. We will, however, use the explicit expressions for $L_{ij}(\hat{\theta}, \hat{p})$ as obtained above.

Substituting $u(\underline{\theta}) = \frac{p}{\theta} t^{p-1}$ in (3.2), the Bayes estimator μ_t^{**} may be similarly obtained.

The corresponding maximum likelihood estimators are

$$\hat{R}_t = \exp\left(-\frac{t^{\hat{p}}}{\hat{\theta}}\right) \quad \text{and} \quad \hat{\mu}_t = \frac{\hat{p}}{\hat{\theta}} t^{\hat{p}-1}$$

where $(\hat{p}, \hat{\theta})$ are the solutions of the likelihood equations

$$\frac{\partial L}{\partial \theta} \equiv \frac{\sum_{i=1}^{n} x_i^p}{\theta^2} - \frac{n}{\theta} = 0,$$

$$\frac{\partial L}{\partial p} \equiv \frac{n}{p} + \sum_{i=1}^{n} (\log x_i) - \frac{\sum_{i=1}^{n} x_i^p (\log x_i)}{\theta} = 0$$

Cohen (1965) and Sinha (1986) have discussed the solutions of the above equations in detail.

Numerical Results

A random sample of size $n = 25$ was generated from the pdf (1.1) with $p = 2$, $\theta = 4$. The mle $\hat{p} = 1.9992$, $\hat{\theta} = 5.4918$. Let SED $(\tilde{\theta})$ = squared-error deviation of an estimator $\tilde{\theta}$ from

the true θ $= (\tilde{\theta} - \theta)^2$.

The entries within the parentheses in the following table represent the corresponding SED.

TABLE 1

*Bayes (Lindley: **; Integration: *) and the MLE* (Λ) *of* R_t *and* μ_t

R_t

t	True R_t	$\hat{R}$	R_t^{**}	R_t^{*}
4	1.8316×10^{-2}	5.4465×10^{-2} (1.3068×10^{-3})	6.6941×10^{-2} (2.3644×10^{-3})	6.7096×10^{-2} (2.3795×10^{-3})
6	1.2341×10^{-4}	1.4361×10^{-3} (1.7230×10^{-6})	3.3802×10^{-3} (1.0607×10^{-5})	5.2130×10^{-3} (2.5904×10^{-5})
8	1.1301×10^{-7}	8.8564×10^{-6} (7.6447×10^{-11})	5.1773×10^{-5} (3.0001×10^{-9})	5.0890×10^{-4} (2.5901×10^{-7})
10	1.3888×10^{-11}	1.2779×10^{-8} (1.6295×10^{-16})	1.7974×10^{-7} (3.2301×10^{-14})	7.4514×10^{-5} (6.0001×10^{-9})
15	3.7234×10^{-25}	1.7595×10^{-18} (3.0958×10^{-36})	1.4105×10^{-16} (1.9895×10^{-32})	2.3063×10^{-6} (5.3190×10^{-12})

μ_t

t	True μ_t	$\hat{\mu}_t$	μ_t^{**}	μ_t^{*}
4	2.0000	1.4545 (2.9757×10^{-1})	1.4596 (2.9203×10^{-1})	1.4545 (2.9214×10^{-1})
6	3.0000	2.1811 (6.7060×10^{-1})	2.1984 (6.4256×10^{-1})	2.2688 (5.3465×10^{-1})
8	4.0000	2.9074 (1.9377)	2.9420 (1.1194)	3.1345 (7.4909×10^{-1})
10	5.0000	3.6336 (1.8670)	3.6894 (1.7177)	4.0516 (8.9946×10^{-1})
15	7.5000	5.4487 (4.2078)	5.5720 (3.7172)	6.5480 (9.5201×10^{-1})

We observe that while the MLE of R_t has uniformly smaller SED than its Bayesian complements, on the basis of the SED, Lindley's estimates are more efficient than those obtained from the marginal posteriors and conversely for μ_t (except at $t = 4$ where the SED of μ_t^{**} is slightly more than that of μ_t^{*}).

4. SIMULATION

One sample does not tell us much. We generated 1000 ($=N$) samples of the same size and the same set of parameters and compared the efficiencies of Bayes estimators with respect to the MLE. Define

Monte Carlo estimate = Average of the N estimates.

Mean-squared-error (MSE) = Average of the squares of deviations of the N estimates from the true parameter.

We report the results in the following table. Entries within parentheses represent the corresponding MSE.

TABLE 2

A Monte Carlo Comparison of the Efficiency of Bayes Estimates/MLE of R_t and μ_t ($N = 1000$, $n = 25$)

R_t

t	True R_t	$\hat{R}_t$	R_t^{**}	R_t^{*}
4	1.8316×10^{-2}	2.1151×10^{-2} (3.7848×10^{-4})	2.8532×10^{-2} (6.3008×10^{-4})	3.0052×10^{-2} (6.2634×10^{-4})
6	1.2341×10^{-4}	6.4249×10^{-4} (2.5331×10^{-6})	1.4777×10^{-3} (1.0104×10^{-5})	2.3580×10^{-3} (1.4831×10^{-5})
8	1.1301×10^{-7}	2.2146×10^{-5} (1.2512×10^{-8})	8.6431×10^{-5} (1.3096×10^{-7})	3.1240×10^{-4} (4.4718×10^{-7})
10	1.3888×10^{-11}	9.6660×10^{-7} (6.3768×10^{-11})	6.1003×10^{-6} (1.8280×10^{-9})	6.3817×10^{-5} (2.7169×10^{-8})
15	3.7234×10^{-25}	4.5141×10^{-10} (4.1733×10^{-17})	8.3989×10^{-9} (1.1573×10^{-14})	3.8199×10^{-6} (1.7082×10^{-10})

			μ_t	
t	True μ_t	$\hat{\mu}_t$	μ_t^{**}	μ_t^*
4	2.0000	2.4353 (1.6812)	2.4455 (1.7012)	2.5072 (2.0189)
6	3.0000	4.0764 (9.5056)	4.1103 (9.6759)	4.4647 (14.3769)
8	4.0000	5.9706 (32.2308)	6.0404 (32.9479)	6.9751 (60.8238)
10	5.0000	8.1067 (83.2362)	8.2245 (85.3723)	10.0988 (194.3710)
15	7.5000	14.4653 (473.6030)	14.7548 (488.8010)	21.0515 (1825.1701)

Table 2 shows that the MLE of both R_t and μ_t has a smaller MSE than their Bayesian counterparts, and compared to the estimates obtained from the posterior distributions, the efficiency of Lindley's estimate increases sharply with time.

5. CONCLUSION

The Monte Carlo study shows that if the MSE is accepted as an index of precision, the MLE of μ_t and R_t are more efficient that their Bayesian complements and compared to the estimates obtained from the marginal posteriors, the efficiency of Lindley's estimates sharply increases with time.

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