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THE USE OF LINEAR PROGRAMMING IN MANPOWER PLANNING

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Concepts of linear programming are used in a discrete renewal model for the development of a graded population, when the transfer rates between the various grades are given and 'wastage' is replaced by new entrants.

The following questions are considered. Which structures can be attained from a given structure, after one or two steps? Which structures can be re-attained after one or two steps? Attention is paid to the possibility of alternative routes, and it is observed that structures attainable after one step are not necessarily attainable after two steps, and vice versa.

Key words: Barycentric coordinates; Convex hull; Linear programming; Manpower planning; Simplex method.

AMS Classification (1980): Primary, 65K05.

El uso de Programación Lineal en Planeación de Fuerza de Trabajo

Se utilizan conceptos de programación lineal en un modelo dicreto de renovación para el desarrollo de una población jerarquizada, cuando son dadas las tasas de transferencia entre los grados y las personas que dejan el sistema son reemplazadas por nuevas entradas.

Se consideran las siguientes cuestiones. ¿Qué estructuras se pueden obtener de una estructura dada, después de uno o dos pasos? ¿Qué estructuras se pueden re-obtener después de uno o dos pasos? Se concede especial atención a la posibilidad de diferentes alternativas, y se observa que las estructuras obtenidas después de un paso no son necesariamente posibles después de dos pasos, y viceversa.

Palabras clave: Coordenadas baricéntricas; Envoltura convexa; Programación lineal; Planeación de fuerza de trabajo; Método simplex.

Clasificación AMS (1980): Primaria, 65K05.

Introduction

We consider the structure of a population which is divided into grades, and its development in a sequence of finite steps. The transition rates between the grades are given and remain fixed in time, and they are such that the total population suffers a loss in numbers, a 'wastage'. However, it is required to keep the total population constant, and the wastage is replaced by new entrants, who join various grades in variable proportions.

Our main interest is the examination of the possibilities of attaining a desired structure from a given one, and of the possibilities of recovering some structure after one or two steps. Our principal tool for this examination is the simplex method of linear programming.

Notation

Let the structure of a population at step t= 0,1,... be described by the column vector $n(t)= (n_1(t),..., n_k(t))^T$, where T indicates transposition, and let $\Sigma_i n_i (t)$, $i=1,...,k$ equal unity for all t. The proportion of entrants at the end of s steps into grade i is denoted $u_i(s - 1)$, and we write for the column vector of these values u(s-1). The matrix of transition rates from grade i into grade j is $P= (p_{ij})$, independent of t, and the wastage vector is

$$(1 - \Sigma_j p_{1j}, ..., 1 - \Sigma_j p_{kj})^T = w^T, \text{ say.}$$

In our examples we shall use the transition matrix for k= 3

$$P= \begin{vmatrix} 0.3 & 0.4 & 0.1 \\ 0.1 & 0.5 & 0.3 \\ 0.2 & 0.1 & 0.4 \end{vmatrix} \text{, so that the wastage vector is } w = \begin{vmatrix} 0.2 \\ 0.1 \\ 0.3 \end{vmatrix} \text{ and } Pw = \begin{vmatrix} 0.13 \\ 0.16 \\ 0.17 \end{vmatrix}$$

We shall also have occasion to use $P^2 = \begin{vmatrix} 0.15 & 0.33 & 0.19 \\ 0.14 & 0.32 & 0.28 \\ 0.15 & 0.17 & 0.21 \end{vmatrix}$.

Attainability

Let us, to begin with, see what happens to the initial structure

$$n(0) = \begin{vmatrix} 0.32 \\ 0.43 \\ 0.25 \end{vmatrix} \quad \text{after one step. We have then} \quad P^T n(0) = \begin{vmatrix} 0.189 \\ 0.368 \\ 0.261 \end{vmatrix}$$

$$\text{The wastage is } (0.2, 0.1, 0.3) \begin{vmatrix} 0.32 \\ 0.43 \\ 0.25 \end{vmatrix} = 1 - (0.189 + 0.368 + 0.261) = 0.182.$$

We do not insist that the new entrants join any particular grade. They join the thrree grades in proportions r_1, r_2, r_3, so that we obain the structure

$$\begin{vmatrix} 0.189 & + & 0.182\, r_1 \\ 0.368 & + & 0.182\, r_2 \\ 0.261 & + & 0.182\, r_3 \end{vmatrix}$$

The r_i are non-negative. A first glance shows that whatever r_i we choose, we can not obtain just any structure after one step, because we must have $n_1(1) \geq 0.189$, $n_2(1) \geq 0.368$, and $n_3(1) \geq 0.261$. Using $r_1 + r_2 + r_3 = 1$, the new structure can be written

$$\begin{vmatrix} 0.371\, r_1 + & 0.189\, r_2 + & 0.189\, r_3 \\ 0.368\, r_1 + & 0.550\, r_2 + & 0.368\, r_3 \\ 0.261\, r_1 + & 0.261\, r_2 + & 0.443\, r_3 \end{vmatrix}$$

In other words, the attainable structures are those in the convex hull of

$$\begin{vmatrix} 0.371 \\ 0.368 \\ 0.261 \end{vmatrix}, \quad \begin{vmatrix} 0.189 \\ 0.550 \\ 0.261 \end{vmatrix}, \quad \begin{vmatrix} 0.189 \\ 0.368 \\ 0.443 \end{vmatrix}$$

These are the points v_1, v_2, v_3 shown, in barycentric coordinates, in figure 1. The subscripts 1, 2, 3, are chosen so as to indicate the single state which new entrants join, to produce the respective structure. In any convex combination, say $r_1 v_1 + r_2 v_2 + r_3 v_3$, the entries into the three grades will also be distributed in proportions r_1, r_2, r_3.

Reverting to our earlier notation, we $u_i(0)= 0.182\ r_i$. We proceed to finding those structures which can be obtained after two steps, from the same initial structure n(0), and with the same transition matrix P as above.

It is at this stage that we shall find the technique of the simplex method of linear programming (see, for instance, Vajda (1967)) useful. We base our analysis on the relationships

$$n(1) = P^T n(0) + u(0) \tag{1}$$

$$n(2) = (P^2)^T n(0) + P^T u(0) + u(1)$$

$$u(0)^T e = n(0)^T w \tag{2}$$

$$u(1)^T e = n(0)^T P w + u(0)^T w$$

where e is the vector $(1,1,1)^T$. We have now

$$n_1(2)= 0.15n_1(0) +0.14n_2(0) +0.15n_3(0) +0.3u_1(0)+0.1u_2(0)+0.2u_3(0)+ u_1(1)$$

$$n_2(2)= 0.33n_1(0) +0.32n_2(0) +0.17n_3(0) +0.4u_1(0)+0.5u_2(0)+0.1u_3(0)+u_2(1)$$

$$n_3(2)= 0.19n_1(0) +0.28n_2(0) +0.21n_3(0) +0.1u_1(0)+0.3u_2(0)+0.4u_3(0)+ u_3(1)$$

$$u_1(0) + u_2(0) + u_3(0) = 0.2n_1(0) + 0.1n_2(0) + 0.3n_3(0)$$

$$u_1(1)+u_2(1)+u_3(1)=0.13n_1(0)+0.16n_2(0)+0.17n_3(0)+0.2u_1(0)+0.1u_1(0)+0.3u_3(0).$$

Given $n_1(0)$, $n_2(0)$, $n_3(0)$, the unknowns are the $n_i(2)$, the $u_i(0)$, and the $u_i(1)$; they must all be non-negative.

We have nine unknowns and five equations. All solution points lie in a convex region in nine-dimensional space, and the vertices of this region can be found by the simplex method.

As a matter of fact, the simplex method was devised to find that vertex, or those vertices, which are optimal, in the sense that they maximise or minimise some linear expression, the "objective function". The answer can be multiple, and if our computer has a code for finding *all* optimal vertices, then we take the objective function $0.x_1 + 0.x_2 + ... + 0.x_n$, when all vertices give it the same optimal value, and the simplex method will then produces all vertices.

With our present assumptions of $n(0) = (0.32, 0.43, 0.25)^T$ and the transition matrix P as given above, the equations read:

$$n_1(2) - 0.3\,u_1(0) - 0.1\,u_2(0) - 0.2\,u_3(0) - u_1(1) = 0.1457$$

$$n_2(2) - 0.4\,u_1(0) - 0.5\,u_2(0) - 0.1\,u_3(0) - u_2(1) = 0.2857$$

$$n_3(2) - 0.1\,u_1(0) - 0.3\,u_2(0) - 0.4\,u_3(0) - u_3(1) = 0.2337$$

$$u_1(0) + u_2(0) + u_3(0) = 0.1820$$

$$u_1(1) + u_2(1) + u_3(1) - 0.2\,u_1(0) - 0.1\,u_2(0) - 0.3\,u_3(0) = 0.1529.$$

As an illustration, we quote the first few steps of the simplex procedure, starting with the tableau

	$u_1(0)$	$u_2(0)$	$u_3(0)$	$u_1(1)$	$u_2(1)$	$u_3(1)$	
$n_1(2)$	-0.3	-0.1	-0.2	-1.0	0.0	0.0	0.1457
$n_2(2)$	-0.4	-0.5	-0.1	0.0	-1.0	0.0	0.2857
$n_3(2)$	-0.1	-0.3	-0.4	0.0	0.0	-1.0	0.2337
s	1.0*	1.0	1.0	0.0	0.0	0.0	0.1820
t	-0.2	-0.1	-0.3	1.0	1.0	1.0	0.1529

We have chosen $n_1(2)$, $n_2(2)$ and $n_3(2)$ to be the basic variables and we introduced two "artificial" variables, s and t, which will have to be discarded before we obtain the final answer.

Choosing as our pivot the entry marked with an asterisk, we have

	$u_2(0)$	$u_3(0)$	$u_1(1)$	$u_2(1)$	$u_3(1)$	
$n_1(2)$	0.2	0.1	-1.0	0.0	0.0	0.2003
$n_2(2)$	-0.1	0.3	0.0	-1.0	0.0	0.3585
$n_3(2)$	-0.2	-0.3	0.0	0.0	-1.0	0.2519
$u_1(0)$	1.0	1.0	0.0	0.0	0.0	0.1820
t	0.1	-0.1	1.0*	1.0	1.0	0.1893

which turns into

	$u_2(0)$	$u_3(0)$	$u_2(1)$	$u_3(1)$	
$n_1(2)$	0.3	0.0	1.0	1.0	0.3896
$n_2(2)$	-0.1	0.3	-1.0	0.0	0.3585
$n_3(2)$	-0.2	-0.3	0.0	-1.0	0.2519
$u_1(0)$	1.0	1.0	0.0	0.0	0.1820
$u_1(1)$	0.1	-0.1	1.0	1.0	0.1893

We have found one vertex. The result means the following transitions:

$$\begin{vmatrix} 0.32 \\ 0.43 \\ 0.25 \end{vmatrix} \rightarrow \begin{vmatrix} 0.189+0.182 \\ 0.368+0 \\ 0.261+0 \end{vmatrix} = \begin{vmatrix} 0.371 \\ 0.368 \\ 0.261 \end{vmatrix} \rightarrow \begin{vmatrix} 0.2003+0.1893 \\ 0.3585+0 \\ 0.2519+0 \end{vmatrix} = \begin{vmatrix} 0.3896 \\ 0.3585 \\ 0.2519 \end{vmatrix}$$

This is, of course, not the only vertex, and proceeding we find all the vertices in nine-dimensional space, as follows:

	v_{11}	v_{12}	v_{13}	v_{21}	v_{22}
$n_1(2)$	0.3896	0.2003	0.2003	0.3350	0.1639
$n_2(2)$	0.3585	0.5478	0.3585	0.3767	0.5478
$n_3(2)$	0.2519	0.2519	0.4412	0.2883	0.2883
$u_1(0)$	0.1820	0.1820	0.1820	0.0	0.0
$u_2(0)$	0.0	0.0	0.0	0.1820	0.1820
$u_3(0)$	0.0	0.0	0.0	0.0	0.0
$u_1(1)$	0.1893	0.0	0.0	0.1711	0.0
$u_2(1)$	0.0	0.1893	0.0	0.0	0.1711
$u_3(1)$	0.0	0.0	0.1893	0.0	0.0

	v_{23}	v_{31}	v_{32}	v_{33}
$n_1(2)$	0.1639	0.3896	0.1821	0.1821
$n_2(2)$	0.3767	0.3039	0.5114	0.3039
$n_3(2)$	0.4594	0.3065	0.3065	0.5140
$u_1(0)$	0.0	0.0	0.0	0.0
$u_2(0)$	0.1820	0.0	0.0	0.0
$u_3(0)$	0.0	0.1820	0.1820	0.1820
$u_1(1)$	0.0	0.2075	0.0	0.0
$u_2(1)$	0.0	0.0	0.2075	0.0
$u_3(1)$	0.1711	0.0	0.0	0.2075

Because our system has five equations, five of the variables of each vertex have positive values, while the other four are zero. We have denoted the vertices v_{ij} in such a way that the first (second) subscript gives the grade which new entrants join after one (two) step(s).

A structure attainable after two steps may be attainable in a variety of ways. For instance, the point $(v_1 + v_2 + v_3) / 3 = (0.25, 0.43, 0.32)^T$ is attainable from n(0) in more than one way. One of them is (v_{11}, v_{12}, v_{13}), and another is (v_{31}, v_{32}, v_{33}) (there are others as well). Thus we have:

$$\begin{vmatrix} 0.32 \\ 0.43 \\ 0.25 \end{vmatrix} \rightarrow v_1 = \begin{vmatrix} 0.371 \\ 0.368 \\ 0.261 \end{vmatrix} \rightarrow \begin{vmatrix} 0.200+0.050 \\ 0.358+0.072 \\ 0.252+0.068 \end{vmatrix} = \begin{vmatrix} 0.25 \\ 0.43 \\ 0.32 \end{vmatrix}$$

and also

$$\begin{vmatrix} 0.32 \\ 0.43 \\ 0.25 \end{vmatrix} \rightarrow v_3 = \begin{vmatrix} 0.189 \\ 0.368 \\ 0.443 \end{vmatrix} \rightarrow \begin{vmatrix} 0.182+0.068 \\ 0.304+0.126 \\ 0.306+0.014 \end{vmatrix} = \begin{vmatrix} 0.25 \\ 0.43 \\ 0.32 \end{vmatrix}$$

n(0) lies also outside the convex hull of the v_{ij}, and is not attainable from itself after two steps. The easiest way of showing arithmetically that this is so, is to use equations (1) and (2), to write $n(0) = n(2) = (0.32, 0.43, 0.25)^T$ and to see whether the equations can be solved for non-negative $u_i(0)$ and $u_j(1)$. We have now

$$0.32 = 0.15(0.32) + 0.14(0.43) + 0.15(0.25) + 0.3u_1(0) + 0.1u_2(0) + 0.2u_3(0) + u_1(1)$$

$$0.43 = 0.33(0.32) + 0.32(0.43) + 0.17(0.25) + 0.4u_1(0) + 0.5u_2(0) + 0.1u_3(0) + u_2(1)$$

$$0.25 = 0.19(0.32) + 0.28(0.43) + 0.21(0.25) + 0.1u_1(0) + 0.3u_2(0) + 0.4u_3(0) + u_3(1)$$

$$0.20(0.32) + 0.10(0.43) + 0.30(0.25) = u_1(0) + u_2(0) + u_3(0)$$

The second equation of the set (2) is now a consequence of those given, by virtue of $n_1 + n_2 + n_3 = 1$. In tableau from, this is

	$u_1(0)$	$u_2(0)$	$u_3(0)$	
$u_1(1)$	0.3	0.1	0.2	0.174
$u_2(1)$	0.4	0.5	0.1	0.144
$u_3(1)$	0.1	0.3	0.4	0.016
s	1.0*	1.0	1.0	0.182.

which turns into

	$u_2(0)$	$u_3(0)$	
$u_1(1)$	-0.2	-0.1	0.120
$u_2(1)$	0.1	-0.3	0.071
$u_3(1)$	0.2	0.3	-0.002
$u_1(0)$	1.0	1.0	0.182

The third row means $u_3(1) + 0.2\, u_2(0) + 0.3\, u_3(0) = -0.002$, and this can not be solved for non-negative $u_3(1)$, $u_2(0)$, $u_3(0)$.

Re-attainability

The question whether n(0) can be re-attained after one or two steps (it can not) suggests an investigation of which initial structures can be thus re-attained.

Such structures must satisfy equations (1) and (2) with n(0)= n(1)= n, say, or with n(0)= n(2)= n, say, respectively, where n is now a variable.

For one step, the equations read now

$$n = P^T n + u(0) \quad \text{and} \quad u(0)^T e = n^T w.$$

The second of these ensures that the total population is again unity, and can therefore be replaced by $n_1 + n_2 + n_3 = 1$.

Once more, the vertices of the region of re-attainable points can be found by linear programming methods. The first tableau reads

	n_1	n_2	n_3	
$u_1(0)$	-0.7	0.1	0.2	0.0
$u_2(0)$	0.4	-0.5	0.1	0.0
$u_3(0)$	0.1	0.3	-0.6	0.0
s	1.0	1.0	1.0	1.0

This time the use of the dual simplex method is indicated. We quote the final result. The vertices are:

	w_1	w_2	w_3
n_1	0.392	0.162	0.193
n_2	0.362	0.541	0.263
n_3	0.246	0.297	0.544
$u_1(0)$	0.189	0.0	0.0
$u_2(0)$	0.0	0.176	0.0
$u_3(0)$	0.0	0.0	0.228

To find the vertices of the region of those points which are re-attainable after two steps we have to solve the problem presented by the tableau

	n_1	n_2	n_3	$u_1(0)$	$u_2(0)$	$u_3(0)$	
$u_1(1)$	-0.85	0.14	0.15	0.3	0.1	0.2	0.0
$u_2(1)$	0.33	-0.68	0.17	0.4	0.5	0.1	0.0
$u_3(1)$	0.19	0.28	-0.79	0.1	0.3	0.4	0.0
s	0.2	0.1	0.3	-1.0	-1.0	-1.0	0.0
t	1.0	1.0	1.0	0.0	0.0	0.0	1.0

The dual Simplex method produces the following answers:

	w_{11}	w_{12}	w_{13}	w_{21}	w_{22}	w_{23}
n_1	0.392	0.196	0.210	0.337	0.162	0.168
n_2	0.362	0.543	0.339	0.376	0.541	0.356
n_3	0.246	0.261	0.451	0.287	0.297	0.476
$u_i(0)$	0.189	0.172	0.212	0.191	0.176	0.212
$u_i(1)$	0.189	0.191	0.200	0.172	0.176	0.181
	*	*		*	*	

	w_{31}	w_{32}	w_{33}
n_1	0.399	0.181	0.193
n_2	0.298	0.505	0.263
n_3	0.303	0.314	0.544
$u_i(0)$	0.200	0.181	0.228
$u_i(1)$	0.212	0.212	0.228
	*		*

The entrants join the grades given respectively by the first and by the second subscript of the w_{ij}. For instance, w_{21} is re-attained after two steps as follows:

$$\left|\begin{matrix}0.337\\0.376\\0.287\end{matrix}\right| \rightarrow \left|\begin{matrix}0.196+0\\0.352+0.191\\0.261+0\end{matrix}\right| = \left|\begin{matrix}0.196\\0.543\\0.261\end{matrix}\right| \rightarrow \left|\begin{matrix}0.165+0.172\\0.376+0\\0.287+0\end{matrix}\right| = \left|\begin{matrix}0.337\\0.376\\0.287\end{matrix}\right|$$

Again, a point re-attainable after two steps may be re-attainable in different ways. For instance, we have seen above how w_{21} was re-attainable. However, it lies also within triangle (w_{11}, w_{12},w_{13}). Indeed

$$\left|\begin{matrix}0.337\\0.376\\0.287\end{matrix}\right| = 0.706 \left|\begin{matrix}0.392\\0.362\\0.246\end{matrix}\right| + 0.102 \left|\begin{matrix}0.196\\0.543\\0.261\end{matrix}\right| + 0.192 \left|\begin{matrix}0.210\\0.339\\0.451\end{matrix}\right|$$

and one possible way of re-attaining w_{21} after two steps, different from the one exhibited earlier, is this:

$$\begin{vmatrix} 0.337 \\ 0.376 \\ 0.287 \end{vmatrix} \rightarrow \begin{vmatrix} 0.196+0.706(0.189)+0.102(0.172)+0.192(0.212) \\ 0.351 \\ 0.261 \end{vmatrix} = \begin{vmatrix} 0.388 \\ 0.351 \\ 0.261 \end{vmatrix}$$

$$\begin{vmatrix} 0.388 \\ 0.351 \\ 0.261 \end{vmatrix} \rightarrow \begin{vmatrix} 0.204+0.706(0.189) \\ 0.357+0.102(0.191) \\ 0.248+0.192(0.200) \end{vmatrix} = \begin{vmatrix} 0.337 \\ 0.376 \\ 0.287 \end{vmatrix}$$

Attainabilities and re-attainabilities after more than two steps can be studied using the same principles as explained above. For more detail, we refer the reader to Vajda (1978).

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