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AN EXTREME - MARKOVIAN - EVOLUTIONARY (EME) SEQUENCE

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The most general sequence, with Gumbel margins, generated by maxima procedures in an auto-regressive way (one step) is defined constructively and its properties obtained; some remarks for statistical estimation are presented.

Key words: Random sequence with Gumbel margins; Dependence functions; Essential and incidental parameters.

AMS Classification (1980): Primary, 60K99; Secondary, 60M05.

Una sucesión extrema evolutiva Markoviana

En este artículo se define constructivamente la sucesión mas general, con marginales Gumbell, generada por procedimientos de máximo en forma auto-regresiva. Se obtienen sus propiedades y se analiza su estimación estadística.

Palabras Clave: Sucesión aleatoria con marginales Gumbell; Funciones de dependencia; Parámetros esenciales y marginales.

Clasificación AMS (1980): Primaria, 60K99; Secundaria, 60M05.

INTRODUCTION

The purpose of this paper is to obtain a model for evolutionary phenomena, through a random sequence with Gumbel margins, model that may, eventually, describe the evolution in time of all natural processes that are, in general, described as IID sequences with Gumbel margins, after some reduction. It extends previous papers by the Tiago de Oliveira (1968, 1972, 1973) in extremal processes and extreme-markovian-stationary (EMS) sequences and processes.

Some hints that can be used for quick statistical decision for EME- sequences are given .

Let us recall that a general Gumbel random variable has the distribution function $\Lambda((x-\lambda)/\delta)$ where $\Lambda(z)=\exp(-e^{-z})$ is the distribution function of a reduced random variable ($\lambda=0, \delta=1$); we have also $\Lambda'(z) = e^{-z}\Lambda(z)$. The first moments are $\mu = \lambda + \gamma\delta$ ($\gamma = 0.57722...$, Euler's constant), $\sigma^2= \pi^2/6.\delta^2 = 1.64493...\delta^2$, $\sigma = \pi/\sqrt{6}.\delta = 1.28825...\delta$, $\beta_1 = \mu_3/\sigma^3 = 1.13955$ ... and $\beta_2 = \mu_4/\sigma^4 = 5.4$; see Gumbel (1958).

An extreme random pair (X,Y) with reduced Gumbel margins has the distribution function $\Lambda(x,y)=\exp\{-(e^{-x}+e^{-y})\,k(y-k)\}=\{\Lambda(x)\,\Lambda(y)\}^{k(y-x)}$ where the dependence function k(.) satisfies some conditions for $\Lambda(.,.)$ to be a distribution function with reduced Gumbel margins; see Tiago de Oliveira (1962/63); in particular we have $\max(1, e^w)/(1+e^w) \leq k(w) \leq 1$, the lower bound corresponding to the diagonal case where we have Prob{Y= X}= 1 (for reduced margins) and k(w)=1 corresponding to independece. The correlation coefficient has the expression

$$\rho = (-6/\pi^2)\int_{(-\infty,+\infty)} \log k(w)\, dw \quad (\geq 0);$$

$\rho= 0$ is equivalent to independence and $\rho= 1$ is equivalent to the diagonal case.

The probability $D(w)= \text{Prob}\{Y - X \leq w\}$, (X,Y) being reduced, has the expression $D(w)= (k'(w)/k(w)) + (e^w/(1+e^w))$, if k'(.) exists; in the independence case (k(w)=1) we have the logistic distribution and in the diagonal case we have $D(w) = H(w)$, where $H(w)$ is the Heaviside jump function $H(w) = 0$ if $w < 0$ and $H(w) = 1$ if $w \geq 0$. Note that $\partial^2\Lambda/\partial x\partial y$ does not exist in all cases as k''(.) does not exist always; this is the case in all stochastic processes (sequences) connected with extremes, as it happens in the EME-sequences.

In particular if $k_\theta(w)= 1-\min(\theta, e^w)/(1+e^w)$ $(0 \leq \theta \leq 1)$ we have

$$D_\theta(w)= 0 \qquad \text{if } w < \log\theta \ (<0) \text{ and}$$
$$D_\theta(w)= (1+(1-\theta)e^{-w})^{-1} \quad \text{if } w \geq \log\theta,$$

with a jump of θ at $w = \log\theta$, the probability of $Y= X+\log\theta$; the correlation coefficient is

$$\rho_\theta = R(\theta) = (-6/\pi^2)\int_{(0,\theta)} \log t/(1-t)\, dt$$

for $\theta= 0$ and $\theta= 1$ we have, obviously, the independence and diagonal cases.

THE GENERATION OF AN EME- SEQUENCE

Let $\{E_t\}$ (t= 0,1,2,...) be a sequence of IID random variables with standard Gumbel margins and $X_0 = \lambda_0 + \delta_0 Z_0$ a Gumbel random variable with parameters (λ_0, δ_0), i.e. Z_0 is a standard Gumbel random variable; Z_0 is independent of $\{E_t\}$. Consider, then, the auto-regressive sequence

$$X_{t+1} = \max(a + bX_t, a' + b'E_t), \quad t = 0,1,2,...$$

where the X_t are supposed to be Gumbel ramdom variables. Let (λ_t, δ_t) be the parameters of X_t, i.e., $X_t = \lambda_t + \delta_t Z_t$ where Z_t are standard Gumbel random variables. The auto-regressive relation can, then, be written as

$$\lambda_{t+1} + \delta_{t+1} Z_{t+1} = \max(a + b(\lambda_t + \delta_t Z_t), a' + b'E_t), \quad t = 0,1,2,...$$

and for $\text{Prob}\{Z_{t+1} \leq x\}$ to be $\Lambda(x)$ we get

$$\delta_{t+1} = b\, \delta_t$$

$$\delta_{t+1} = b'$$

$$\exp((a + b\,\lambda_t - \lambda_{t+1}) / b\,\delta_t) + \exp((a' - \lambda_{t+1}) / b') = 1$$

so that $\delta_t = \text{const} = \delta_0$, $b = 1$ and $\exp(\lambda_{t+1} / \delta_0) = \exp((a + \lambda_t) / \delta_0) + \exp(a' / \delta_0)$.

The auto-regressive relation takes, then, the form

$$X_{t+1} = \max(a + X_t, a' + \delta_0 E_t).$$

Consider now the "padronized" sequence $Y_t = (X_t - \lambda_0) / \delta_0$ $(Y_0 = Z_0)$ with $Z_t = (X_t - \lambda_t) / \delta_t = Y_t - (\lambda_t - \lambda_0) / \delta_0 = Y_t - \eta_t$ $(\eta_t = (\lambda_t - \lambda_0) / \delta_0,\ \eta_0 = 0)$. The auto-regressive equation for Y_t is

$$Y_{t+1} = \max(a_0 + Y_t, a'_0 + E_t), \quad t = 0,1,2,...$$

with $a_0 = a / \delta_0$, $a'_0 = (a' - \lambda_0) / \delta_0$ and the relation for $\{\lambda_t\}$ takes the form

$$\exp(\eta_{t+1}) = \exp(a_0) \exp(\eta_t) + \exp(a_0').$$

Summarizing: the EME- sequence $\{X_t\}$, with X_0 with parameters (λ_0, δ_0), satisfies the relation

$$X_{t+1} = \max(a_0 \delta_0 + X_t,\ \lambda_0 + a_0' \delta_0 + \delta_0 E_t)$$

and the padronized sequence $\{Y_t\}$ verifies

$$Y_{t+1} = \max(a_0 + Y_t,\ a_0' + E_t);$$

thus (a_0, a_0') are the *essential* parameters and (λ_0, δ_0) are *incidental* parameters; X_t is reconstituted by the relation $X_t = \lambda_0 + \delta_0 Y_t$; as $Y_t = Z_t + \eta_t$ the parameters of X_t (as margins) are $(\lambda_0 + \delta_0 \eta_t,\ \delta_0)$.

Let us study, now, the basic equation

$$\exp(\eta_{t+1}) = \exp(a_0) \exp(\eta_t) + \exp(a_0'), \quad \text{with } \eta_0 = 0:$$

if $\exp(a_0) = 1 \quad (a_0 = 0)$ we have $\exp(\eta_t) = 1 + \exp(a_0')\, t$;

if $\exp(a_0) \neq 1 \quad (a_0 \neq 0)$ we get

$$\exp(\eta_t) = [\exp(a_0') / (1 - \exp(a_0))] + [1 - (\exp(a_0') / (1 - \exp(a_0))] \exp(a_0 t);$$

remark that $\lim_{a_0 \to 1} \exp(\eta_t) = 1 + \exp(a_0')\, t$, the expression of $\exp(\eta_t)$ for $a_0 = 0$.

The condition of stationarity imposes $\eta_{t+1} = \eta_t = \ldots = \eta_0 = 0$ so that $\exp(a_0) + \exp(a_0') = 1$; this is the condition for stationarity obtained in Tiago de Oliveira (1972); $\theta = \exp(a_0)$ was the dependence parameter introduced there.

Let us consider the monotonicity behaviour. As $M(Y_t) = M(\eta_t + Z_t) = \gamma + \eta_t$ we see that Y_t (and X_t) is increasing, stable or decreasing in mean acoording to $\eta_{t+1} > \eta_t$, $\eta_{t+1} = \eta_t$ or $\eta_{t+1} < \eta_t$, i.e., according to $\exp(a_0) + \exp(a_0') > 1$, $\exp(a_0) + \exp(a_0') = 1$ (stationarity) or $\exp(a_0) + \exp(a_0') < 1$. Remark that if $a_0 = 0$ we do not have the last behaviour and we get constancy only if $a_0 = 0$ and $a_0' = -\infty$, i.e., $Y_{t+1} = Y_t$.

Compute now $\text{Prob}\{Y_{t+1} > Y_t\}$. If $a_0 > 0$ it is immediate that $Y_{t+1} > a_0 + Y_t$ so that $\text{Prob}\{Y_{t+1} > Y_t\} = 1$: the EME-sequence is increasing with probability one and, so, the method of "ups and downs" considered in Tiago de Oliveira (1972) for the estimation of $\theta = \exp(a_0)$ (<1) can not be used.

For $a_0 = 0$ we have $\mathrm{Prob}\{Y_{t+1} > Y_t\} = (1 + \exp(-a_0') + t)^{-1}$ decreasing with t; as $\mathrm{Prob}\{Y_{t+1} = Y_t\} = 1 - \mathrm{Prob}\{Y_{t+1} > Y_t\} > 1$ we see that the sequence stabilizes asymptotically.

When $a_0 < 0$ we have

$$\mathrm{Prob}\{Y_{t+1} > Y_t\} = 1 - \mathrm{Prob}\{Y_{t+1} \leq Y_t\}$$

$$= (1 - \exp(a_0)) \exp(a_0') \,/\, [(2 - \exp(a_0)) \exp(a_0') + (1 - \exp(a_0) - \exp(a_0')) \exp(a_0 t)]$$

which takes the value $\exp(a_0') / (1 + \exp(a_0'))$ at $t = 0$ and converges to $(1 - \exp(a_0)) / (2 - \exp(a_0))$ as $t \to \infty$, increasing if $\exp(a_0) + \exp(a_0') < 1$, in a stable way if $\exp(a_0) + \exp(a_0') = 1$ (stability) and decreasing if $\exp(a_0) + \exp(a_0') > 1$.

Then the sequence increases in median ($\mathrm{Prob}\{Y_{t+1} - Y_t\} > 1/2$) always if $a_0 > 0$, when $t \leq 1 - \exp(-a_0')$ if $a_0 = 0$ and $a'_0 > 0$ but never when $a_0' < 0$ and when $\exp(-a_0 t) < ((\exp(a_0) + \exp(a_0') - 1) \exp(-a_0) \exp(-a_0')$ if $(\exp(a_0) + \exp(a_0') > 1$ but never if $(\exp(a_0) + \exp(a_0') \leq 1$.

Let us consider now the bivariate structure of the EME-sequence, the multivariate one being an easy extension.

Taking $s < t$ we have

$$Y_t = \max(a_0 + Y_{t-1}, a'_0 + E_{t-1}) =$$

$$= \max(2a_0 + Y_{t-2}, a_0 + a'_0 + E_{t-2}, a'_0 + E_{t-1}) = \ldots =$$

$$= \max((t-s)a_0 + Y_s, a'_0 + \max_{p=1,t-s}((p-1)a_0 + E_{t-p})).$$

Then

$$\mathrm{Prob}\{Y_s \leq x, Y_t \leq y\} = \Lambda(\min(x, y - (t-s)a_0 - \eta_s)) \times$$

$$\prod_{p=1,t-s} \Lambda(y - a'_0 - (p-1)a_0) \quad \text{if } a_0 \neq 0$$

and $\mathrm{Prob}\{Y_s \leq x, Y_t \leq y\} = \exp\{-\max(e^{-x}, e^{-y}) \exp(\eta_s) - \exp(a_0')(t-s)\, e^{-y}\}$ if $a_0 = 0$.

To obtain the correlation structure we need the dependence function $K_{s,t}(.)$, associated with the bivariate structure of $(Z_s, Z_t) = (Y_s - \eta_s, Y_t - \eta_t)$. We get from

$$\text{Prob}\{Z_s \le x, Z_t \le y\} = \text{Prob}\{Y_s \le x + \eta_s, Y_t \le y + \eta_t\}$$

$$= (\Lambda(x)\,\Lambda(y))^{\delta},$$

with $\delta = k_{s,t}\,(y - x)$, after easy calculations,

$$k_{s,t}\,(w) = 1 - [\ \min(\exp((t-s)\,a_0 - \eta_t + \eta_s),\ e^w)\,/\,(1 + e^w)\ ],$$

Where we have $a_0 = 0$ or $a_0 \ne 0$, with the corresponding expressions of η_t. The correlation coefficient is, then,

$$\rho_{s,t} = R[\ \exp((t-s)\,a_0 - \eta_t + \eta_s)\ ],$$

as said before, with

$$R(\theta) = (-6/\pi^2) \int_{(0,\theta)} \log t\,/(1-t)\,dt;$$

remark that $0 \le \exp((t-s)\,a_0 - \eta_t + \eta_s) \le 1$. For $a_0 = 0$ we get

$$\rho_{s,t} = R[\ (1 + \exp(a_0')s)\,/\,(1 + \exp(a_0')t)\ ]$$

and for $a_0 \ne 0$ we have

$$\rho_{s,t} = R[\ (1 - \exp(a_0) - \exp(a_0') + \exp(a_0' - a_0 s))$$

$$/\,(1 - \exp(a_0) - \exp(a_0') + \exp(a_0' - a_0 t))\ ].$$

If we have an EMS- sequence, i.e. if $\exp(a_0) + \exp(a_0') = 1$ we get $\rho_{s,t} = R[\exp(a_0(t-s))]$ as obtained in Tiago de Oliveira (1972) where $\theta = \exp(a_0)$ as said. Evidently independence, corresponding to $a_0 = -\infty$, $a_0' = 0$ leads to $\rho_{s,t} = R(0) = 0$, as expected.

SOME BASIC RESULTS

Let us prove some propositions which are important to analyse the behaviour of EME-sequences.

A first result is that $Y_t / t \to^{m.s.} \max(a_0, 0)$ and $Y_{t-1} - Y_t \to^{m.s.} a_0$ if $a_0 > 0$ as $t \to \infty$. In fact as $Y_t = \eta_t + Z_t$ we have

$$M(Y_t / t) = (\gamma + \eta_t) / t \to \max(a_0, 0) \text{ and}$$

$$V(Y_t / t) = (\pi^2 / 6) / t^2 \to 0 \text{ and also}$$

$$M(Y_{t+1} - Y_t) = \eta_{t+1} - \eta_t \to a_0 \text{ if } a_0 > 0 \text{ and}$$

$$V(Y_{t+1} - Y_t) = \pi^2 / 3. (1 - \rho_{t,t+1}) \to 0.$$

As $Y_{t+1} - Y_t = a_0$ (or $a_0' + E_t \leq a_0 + Y_t$) with probability

$$\exp(a_0) \exp(\eta_t) / [\exp(a_0') + \exp(a_0) \exp(\eta_t)] \to \min(1, \exp(a_0)) =$$

$$= \exp(\min(a_0, 0))$$

and $Y_{t+1} - Y_t > a_0$ with complementary probability it seems natural to study the statistic $A_n = \min_{1,n} (Y_t - Y_{t-1})$.

Let us denote by $Q_n(\Lambda_1,..., \Lambda_n) = \text{Prob}\{Y_1 - Y_0 > a_0 + \Lambda_1,..., Y_n - Y_{n-1} > a_0 + \Lambda_n\}$ for $\Lambda_i \geq 0$. We see easily that $D_n = \{Y_1 - Y_0 > a_0 + \Lambda_1,..., Y_n - Y_{n-1} > a_0 + \Lambda_n\}$, as $Y_t = a_0' + E_t$, is equivalent to

$$D_n' = \{a_0' + E_1 > a_0 + \Lambda_1 + Y_0,\ E_2 > a_0 + \Lambda_2 + E_1, ..., E_n > a_0 + \Lambda_n + E_{n-1}\}$$

and we get

$$Q_n(\Lambda_1,..., \Lambda_n) = G_n[\exp(a_0 + \Lambda_1 - a_0'), \exp(a_0 + \Lambda_2), ..., \exp(a_0 + \Lambda_n)]$$

where, with $B = \{t_0 > \varphi_1 t_1, ..., t_{n-1} > \varphi_n t_n\}$,

$$G_n(\varphi_1,..., \varphi_n) = \int_B \exp(-(t_0 + t_1 + ... + t_n))\, dt_0\, dt_1 ... dt_n$$

which satisfies the relation

$$G_n(\varphi_1,\dots,\varphi_n) = (1+\varphi_1)^{-1} G_{n-1}((1+\varphi_1)\varphi_2, \varphi_3,\dots,\varphi_n).$$

Then $Q_n(\Lambda,\dots,\Lambda)=$

$$(1+\exp(a_0+\Lambda - a_0'))^{-1} Q_{n-1}[(1+\exp(a_0+\Lambda - a_0'))\exp(a_0+\Lambda),\dots,\exp(a_0+\Lambda)]$$

which tends to $\underline{0}$ as $n \to \infty$. A_n is, thus an estimator of a_0.

Other results can give hints for statistical estimation and describe the behaviour of the Y_t. It is easy to show that, as

$$M(Y_{t+1} - Y_t - a_0) \to^{m.s.} -\min(a_0, 0), \text{ that } \Sigma_{0,n-1} M(Y_{t+1} - Y_t - a_0) = Y_n - Y_0 - n\, a_0$$

$$\to \log[1 - (\exp(a_0)/1 - \exp(a_0))] \text{ if } a_0 > 0 \text{ and}$$

$$\Sigma_{0,n-1} M(Y_{t+1} - Y_t) \to a_0' - \log[1 - \exp(a_0)] \text{ if } a_0 < 0.$$

Also we have, if $a_0 \leq 0$,

$$\text{Prob}\{\max(Y_0,\dots,Y_n) \leq x\}$$

$$= \text{Prob}\{\max(Y_0, a_0'+E_0,\dots, a_0'+E_{n-1}) \leq (x)\}$$

$$= \Lambda(x)\,\Lambda^{n-1}(x - a_0') = \exp\{-e^{-x}[1 + (n-1)\exp(a_0')]\}$$

and thus $\text{Prob}\{\max(Y_0,Y_1,\dots,Y_n) - \log n \leq x\} \to \Lambda(x - a_0')$.

SOME REMARKS ON STATISTICAL DECISION

Although we did not attain sufficient results for statistical decision some remarks can be made.

It seems natural to split statistical decision for the EME-sequences in two steps: statistical decision concerning the essential parameters (a_0, a_0') and, after, supposing, (a_0, a_0') known to estimate (λ_0, δ_0) by the least-squares method. In principle, the estimators of (a_0, a_0') must be independent of (λ_0, δ_0) and the ones of the incidental parameters to be quasi-linear, i.e., such that $\lambda_0^*(\alpha + \beta X_t) = \alpha + \beta\lambda_0^*(X_t)$ and $\delta_0^*(\alpha + \beta X_t) = \beta\delta_0^*(X_t)$ for $-\infty < \alpha < +\infty$, $0 < \beta < +\infty$, as happens with the least-squares method; see Cramér (1946) and Silvey (1975).

Let us suppose, now, we are dealing with the "padronized sequence" $\{Y_t\}$.

A test of constancy ($a_0 = 0$, $a_0' = -\infty$) is not necessary; to devise a test of independence ($a_0 = -\infty$, $a_0' = 0$) and of stationarity is very important.

An important situation is that $a_0 > 0$. As $Y_t / t \to^{m.s.} a_0 (> 0)$ a natural region for deciding $a_0 > 0$ is to accept this hypothesis if $\{X(t) > A(t)\}$, which is also the Neyman- Pearson test of $\lambda > 0$ againt $\lambda \le 0$ for the distribution $\Lambda(x - \lambda)$. Recall that $\eta_t - a_0 t \to \log[1 + (\exp(a_0') / (\exp(a_0) - 1))]$ if $a_0 > 0$, $\eta_t - \log t \to a_0'$ if $a_0 = 0$ and $\eta_t \to a_0' - \log(1 - \exp(a_0))$ if $a_0 < 0$; η_t increases linearly with t if $a_0 > 0$, logarithmically if $a_0 = 0$ and converges to a constant if $a_0 < 0$.

A(t) can be defined by imposing $\text{Prob}\{Y_t \le A(t) \mid a_0 = 0\} = \alpha$ or

$$\Lambda[A(t) - \log(1 + \exp(a_0')\, t] = \alpha \quad \text{or } A(t) = \log(1 + \exp(a_0')\, t) - \log(-\log \alpha)$$

which depends, yet, as on the fixation of a value a_0'.

A final remark: if $a_0 > 0$, as $Y_{t+1} \ge Y_t$ we have $Y_t \ge Y_0 + a_0 t$ so that after few steps (depending on Y_0, a_0, a_0') we will have practicallly always $Y_{t+1} = a_0 + Y_t$ because $\text{Prob}(a_0' + E_t > Y_0 + a_0 t) = \exp(a_0') / (\exp(a_0') + \exp(a_0 t)) \to 0$ very quicly. In pratice, a_0' can be estimated only by the values of $\{Y_t\}$.

The problem of statistical decision for EME- sequences is yet an open problem, although the results given here may be helpful in a first step.

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