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NEW LIGHT ON THE THEOREM OF PERRON

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We prove that the principal eigenvector of a positive matrix represents the relative dominance of its rows or ranking of alternatives in a decision represented by the rows of a pairwise comparison matrix.

Key words: Consistency; Decision; Dominance; Eigenvalue; Rank preservation.

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Nueva luz sobre el Teorema de Perron

Se demuestra que el autovector de una matriz positiva representa la dominación relativa de las filas de la matriz o del orden de las alternativas en una decisión donde las preferencias son representadas mediante las filas de una matriz de comparaciones en pares.

Palabras Clave: Consistencia; Decisión; Dominancia; Autovalor; Preservación del Orden.

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If $A= (a_{ij})$, $a_{ij} \geq 0$, $i,j= 1,...,n$, (Perron, 1907) proved that A has a real positive maximun eigenvalue λ_{max} (called the principal eigenvalue of A) that is unique and $\lambda_{max} > |\lambda_k|$ for the remaining eigenvalues of A. Furthermore the principal eigenvector $w= (w_1,...,w_n)$ that is a solution of $Aw= \lambda_{max}w$ is unique to within a multiplicative constant and $w_i > 0$, $i= 1,...,n$.

Perron's result has found wide applications in practice. Here we give a result which highlights the significance of the principal eigenvector.

Suppose we wish to rate three alternative teachers A, B, and C according to their excellence in teaching.

We enter our evaluation as in the following matrix:

$$\begin{array}{c c} & \begin{array}{ccc} A & B & C \end{array} \\ \begin{array}{c} A \\ B \\ C \end{array} & \begin{pmatrix} 1 & 7 & 5 \\ 1/7 & 1 & 3 \\ 1/5 & 1/3 & 1 \end{pmatrix} \end{array}$$

This matrix gives the pairwise dominance of the alternative in the row over that in the column. For example, teacher B is rated to be 3 times better than teacher C which is the entry in the (2,3) position. There is no loss in generality in using the reciprocal relation $a_{ji} = 1 / a_{ij}$. From this pairwise comparison matrix we wish to derive a scale of relative standing for A, B and C. At first one may think that this is given by adding the components of each row and normalizing the result. This is only true if the matrix is consistent ($a_{ij}\, a_{jk} = a_{ik}$, $i,j,k = 1,\dots,n$) for then the matrix has unit rank and any row is a multiple of a single row. The above matrix is inconsistent. For example $a_{23} = 3 \neq a_{13} / a_{12} = 5 / 7$. Note that consistency implies the reciprocal relation but not conversely. Let us now examine the general case.

There is a natural way to derive the rank order of a set of alternatives from a pairwise comparison matrix A (Saaty). The rank order of each alternative is the relative proportion of its dominance over the other alternatives. This is obtained by adding the elements in each row in A and dividing by the total over all the rows. However A only captures the dominance of one alternative over each other in one step. But an alternative can dominate a second by first dominating a third alternative and then the third dominates the second. Thus, the first alternative dominates the second in two steps. It is known that the result for dominance in two steps is obtained by squaring the pairwise comparison matrix. Similarly dominance can occur in three steps, four steps and so on, the value of each obtained by raising the matrix to the corresponding power. The rank order of an alternative is the sum of the relative values for dominance in one step, two steps and so on averaged over the number of steps. We show below that when we take this infinite series and calculate its limiting value we obtain the principal right eigenvector of the matrix A.

Definition1: Let the nodes of a directed graph G be denoted by 1,2,...,n. With every directed arc x_{ij} from node i to node j, we associate a nonnegative number, $a_{ij} > 0$ called the *intensity* of the arc. (Loops and multiple arcs are allowed.)

Definition 2: A *walk* in a directed graph is an alternating sequence of nodes and arcs such that each node is the target of the arc in the sequence preceding it and the source of the arc following it. Both endpoints of each are on the sequence. The *length* of a walk is the number of arcs in its sequence. A walk of length k will be called a "k-walk.".

Definition 3: The *intensity of a walk* of length k from node i to node j is the product of the intensities of the arcs in the walk.

Definition 4: The *total intensity* of all k-walks from node i to node j is the sum of the intensities of the walks.

Definition 5: Given a directed graph D, the *intensity-incidence* matrix $A = (a_{ij})$ is defined as the matrix whose entries are given by a_{ij} for all i and j.

The following is known in graph theory (Harary, p.203).

Lemma: The (i,j) entry, $a_{ij}^{(k)}$, of A^k is the total intensity of k-walks from node i to node j.

Definition 6: The dominance of an alternative along all walks of length $k \leq m$ is given by

$$m^{-1} \Sigma_k A^k e / e^T A^k e, \quad k=1,...,m \quad e=(1,1,...,1)$$

Theorem: The dominance of an alternative along all walks k, as $k \to \infty$ is given by the solution of the eigenvalue problem $Aw = \lambda_{max} w$.

Proof: Let

$$s_k = A^k e / e^T A^k e$$

and $t_m = m^{-1} \Sigma_k s_k$; $k=1,...,m$.

Note that $\lim_{m \to \infty} t_m < \infty$. This is a consequence of a theorem due to G.H. Hardy (1949) which gives necessary and sufficient conditions for a transformation of a convergent sequence to also be convergent. Let T be such a transformation mapping

$(s_1,\dots,s_m) \rightarrow t_m = \Sigma_k \, c_{m,k} \, s_k.$

T is regular if $t_m \rightarrow s$ as $m \rightarrow \infty$ whenever $s_k \rightarrow s$ as $k \rightarrow \infty$. It is known that T is regular if and only if the following conditions hold:

(1) $\Sigma_k |c_{m,k}| < H$ (independent of m), $k = 1,\dots,\infty$

(2) $c_{m,k} \rightarrow \delta_k$ for each k, when $m \rightarrow \infty$,

(3) $\Sigma_k c_{m,k} \rightarrow \delta$ when $m \rightarrow \infty$, $k = 1,\dots,\infty$

(4) $\delta_k = 0$ for each k,

(5) $\delta = 1$

Here, $c_{m,k} = \{ m^{-1}$ for $1 \le k \le m$; 0 for $k > m \}$

Thus, we have

(1) $\Sigma_k |c_{m,k}| = \Sigma_k |1/m| = 1$, $k = 1,\dots,\infty$

(2) $c_{m,k} = 1/m \rightarrow 0$ as $m \rightarrow \infty$ Hence (4) $\delta_k = 0$ for each k,

(3) $\Sigma_k c_{m,k} = \Sigma_k 1/m = 1$ and hence (5) $\delta = 1$, $k = 1,\dots,\infty$

It follows that T is regular. Since $s_k = A^k e \,/\, e^T A^k e \rightarrow w$ as $k \rightarrow \infty$ (Perron, 1907) where w is the principal right eigenvector of A, we have

$t_m = m^{-1} \Sigma_k A^k e \,/\, e^T A^k e \rightarrow w$ as $m \rightarrow \infty$.

The eigenvector shows bias by ranking alternatives according to dominance. Here bias is a desirable property!

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