HOMENAJE AL PROFESOR SIXTO RIOS
TRABAJOS DE ESTADISTICA Y DE INVESTIGACION OPERATIVA
Vol. 36, Núm. 3, 1985, pp. 227 a 231

A SIMPLE PROOF OF FISHBURN'S MOMENTS THEOREM.

M.L. Martínez

Departamento de Estadística. Facultad de Ciencias.
Universidad de Málaga. Campus de Teatinos s/n. 29071 Málaga.

We derive a necessary condition for stochastic dominance of any order based on the Laplace transform of probability measures on $[0,\infty)$ from which it follows easily Fishburn's theorem on the lexicographic order of the moments.

Key words: Stochastic dominance; Moments of distributions; Laplace transform.

AMS Classification (1980): Primary, 60E05; Secondary, 60E10.

Una demostración sencilla del Teorema de los Momentos de Fishburn

Se da una condición necesaria para la dominancia estocástica de cualquier orden basada en la transformada de Laplace de medidas de probabilidad sobre $[0,\infty]$ de la que se deduce fácilmente el teorema de Fishburn sobre el orden lexicográfico de los momentos.

Palabras Clave: Dominancia estocástica; Momentos de distribuciones; Transformada de Laplace.

Clasificación AMS (1980): Primaria, 60E05; Secundaria, 60E10.

Let $\mathbf{F}([0,\infty))$ be the set of all probability distribution functions on the positive real line. For any $F \in \mathbf{F}([0,\infty))$ and n= 1,2,... define

$$F^1 = F, \quad F^{n+1}(x) = \int_{[0,x)} F^n(y)\, dy.$$

Borch (1975) and Fishburn (1976) define the nth degree stochastic dominance relation by

$$F \geq_n G \quad \text{if and only if} \quad G^n(x) \geq F^n(x) \quad \text{for all } x \in [0,\infty).$$

These relations are shown to be partial orderings on the set $\mathbf{F}([0,\infty))$ and, consequently (see, e.g., Fishburn (1976) and Girón and Martínez (1983)) can be described in terms of classes of utility functions. In particular

$$F \geq_n G \text{ if and only if } \int u\, dF \geq \int u\, dG \text{ for all } u \in U_n,$$

where

$$U_1 = \{u; \text{ is continuous, increasing and bounded in } [0,\infty)\}$$

$$U_2 = \{u \in U_1; \ u \text{ is concave}\}$$

$$U_3 = \{u \in U_2; \ -u' \in U_2\}$$

and, in general

$$U_n = \{u \in U_{n-1}; \ -u' \in U_{n-1}\}.$$

Explicit characterization of the extremal rays of these classes can be found in Girón and Martínez (1982) and in Girón and Martínez (1985) as well as the corresponding ones for distributions with support in [0,b].

Let U_d denote the class of decreasing absolute risk aversion (DARA) utility functions, that is,

$$U_d = \{u \in U_2; \ r = -u'' / u' \text{ is decreasing}\}.$$

DARA stochastic dominance was defined by Vickson (1975) as

$$F \geq_d G \quad \text{iif} \int u\, dF \geq \int u\, dG \quad \text{for all } u \in U_d.$$

Note that $U_1 \supset U_2 \supset \ldots U_n \supset \ldots$ and their intersection $U_\infty = \cap_n U_n$; $n=1,2,\ldots,\infty$ is non-empty (see the difference with the continua of utility classes considered by Fishburn (1976), where U_∞ is essentialy empty). It is worthwhile remarking that a large class of bounded DARA utility functions belong to U_∞ (in particular the constant risk aversion utility functions and their mixtures).

For any $F \in \mathbf{F}([0,\infty))$ let $\mu_F^0 = 1$, and for any n= 1,2,..., let μ_F^n denote the nth moment of F about the origin.

Finally, let $L_F(s)$, $s \geq 0$, denote the Laplace transform of F, i.e.,

$$L_F(s) = \int_{[0,\infty)} e^{-sx}\, dF(x). \tag{1}$$

Our main result from which Fishburn's theorem follows easily may be stated as

Theorem 1. If either $F \geq_n G$ for some n or $F \geq_d G$, then $L_F(s) \leq L_G(s)$ for every $s \geq 0$. If either $F >_n G$ for some n or $F >_d G$, then $L_F(s) < L_G(s)$ for every $s>0$.

Proof. For every $s \geq 0$, the function $u_s(x) = - e^{-sx}$ belongs to U_∞ and, therefore, to U_n for every n and to U_d. This implies that if either $F \geq_n G$ or $F \geq_d G$, then

$$\int u_s(x)\, dF(x) \geq \int u_s(x)\, dG(x)$$

i.e., $L_F(s) \leq L_G(s)$.

Note that if strict dominance is assumed then $L_F(s) < L_G(s)$ for every $s > 0$.

An alternative proof is based on the following representation of the Laplace transform obtained by integrating by parts formula (1) sucessively

$$L_F(s) = s^{n-1} \int_{[0,\infty)} e^{-sx}\, dF^n(x) \tag{2}$$

From (2)

$$L_G(s) - L_F(s) = s^{n-1} \int_{[0,\infty)} e^{-sx}\, d(\, G^n(x) - F^n(x)\,)$$

and the theorem follows easily.

Note. If we define $F \geq_\infty G$ iif $\int u\, dF \geq \int u\, dG$ for every $u \in U_\infty$, then from theorem 1 and Bernstein's theorem on completely monotone functions we obtain that $F \geq_\infty G$ iif $L_F(s) \leq L_G(s)$ for every $s \geq 0$.

The next theorem given by Fishburn (1980) and recently generalized by O'Brien (1984) follows from theorem 1.

Theorem 2. Let $F, G \in \mathbf{F}([0,\infty))$ be distribution functions for which μ_F^n and μ_G^n exist. If either $F >_n G$ or $F >_d G$ then for some k, with $1 \leq k \leq n$, $\mu_F^i = \mu_G^i$ ($i=0,1,\ldots,k-1$) and $(-1)^k \mu_F^k < (-1)^k \mu_G^k$.

Proof. As the moments through order n of F and G are finite, their Laplace transforms can be expanded as Taylor series in a neighbourhood of the origin, as follows

$$L_F(s) = 1 - \mu_F^1 s + 1/2\ \mu_F^2 s^2 + \dots + (-1)^n/n!\ \mu_F^n s^n + o_1(|s|^n)$$

and

$$L_G(s) = 1 - \mu_G^1 s + 1/2\ \mu_G^2 s^2 + \dots + (-1)^n/n!\ \mu_G^n s^n + o_2(|s|^n)$$

From this we have

$$L_G(s)-L_F(s) = -(\mu_G^1 - \mu_F^1)s + 1/2(\mu_G^2 - \mu_F^2)s^2 + \dots + (-1)^n/n!(\mu_G^n - \mu_F^n)s^n + o(|s|^n).$$

From the preceding theorem and the hypothesis $F >_n G$ or $F >_d G$, it follows that $L_G(s) - L_F(s) > 0$ for every $s > 0$. As $\lim_{s \downarrow 0} o(|s|^n) / s^n = 0$, we can find a neighbourhood of the origin such that for every $s \in (0, \varepsilon)$

$$\Sigma_i ((-1)^i / i!)(\mu_G^i - \mu_F^i)\ s^i > 0 \quad ;\ i = 0,1,\dots,n \qquad (3)$$

From this relation the theorem follows quickly. For suppose that $\mu_F^i = \mu_G^i$ $(i=0,\dots,k-1)$ for some $k \le n$. This is true for k=1. If we divide (3) by s^k and take limits when $s \to 0$ we get $(-1)^k \mu_F^k \le (-1)^k \mu_G^k$. If the inequality is strict we have finished; otherwise we proceed to the next k+1. Obviously not all the remaining μ_F^j and μ_G^j are equal, for then, the left hand side of equation (3) would be 0.

ACKNOWLEDGMENT

I want to thank Prof. F.J. Girón for suggesting the idea of the proof of theorem 1.

REFERENCES

BORCH, K.(1975). Utility and stochastic dominance. In *Expected Utility Hypotheses and the Allais Paradox.* M. Allais and O. Hagen (eds.). Reidel Pub. Comp. 193-201.

FISHBURN, P.C.(1976). Continua of stochastic dominance relations for bounded probability distributions. *J. Math. Economic.* **3**, 295-311.

FISHBURN, P.C(1980). Stochastic dominance and moments of distributions. *Math. Oper. Res*., 94-100.

GIRON, F.J. and MARTINEZ, M.L.(1982). Caracterización de las funciones extremales asociadas a las dominancias estocásticas de orden k. *Actas IX Jor. Mat. Hispano-Lusas ,Vol*. **2**, 645-648.

GIRON, F.J. and MARTINEZ, M.L.(1983). Ciertos resultados generales sobre dominancia estocástica y utilidad esperada. *Actas XIII Jor. Est. Inv.Oper. Vol.* **1**, 128-135.

GIRON, F.J. and MARTINEZ, M.L.(1985). A characterization of nth degree stochastic dominance. *Tech . Report nº 7. Department of Statistics.* University of Málaga.

O'BRIEN, G.L.(1984). Stochastic dominance and moment inequalities. *Math .Oper. Res*., **9**, 475-477.

VICKSON, R.G. (1984). Stochastic dominance tests for decreasing absolute risk aversion I: discrete random variables. Manag. Scien. Appl. Ser., **21**, 1438-1446.