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RICCATI'S DIFFERENTIAL EQUATION IN BIRTH-DEATH PROCESSES

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This note reviews the occurrence of Riccati's equation in three birth-death type processes, and outlines their solutions.

Key words: Riccati equation, Birth-death process, Parity, Dependent process, Two-sex population model.

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Ecuación diferencial de Riccati en procesos de nacimiento-muerte

Este trabajo analiza la ocurrencia de la ecuación de Riccati en tres procesos de nacimiento-muerte y esquematiza sus soluciones.

Palabras clave: Ecuación de Riccati; Procesos de nacimiento-muerte, Paridad, Modelo de población con los sexos.

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The structure of the partial differential equation for the probability generating function (p.g.f.) of processes of the birth-death type is such that its solution often involves a Riccati equation. It is interesting to consider three known cases where such an equation arises, and outline the solutions obtained.

1. THE BIRTH-DEATH PROCESS WITH TIME-DEPENDENT PARAMETERS.

Suppose that a linear birth-death process $X(t)$ has the birth and death rates $\lambda(t)$, $\mu(t)$ respectively, where $t \geq 0$ denotes time. Then, if the p.g.f. of the process is

$$\Phi(s;0,t) = \Sigma_n P_{1n}(0,t)\, s^n, \qquad |s| \leq 1,$$

where $P_{1n}(0,t) = \Pr\{X(t)=n|X(0)=1\}$, this is known to satisfy the partial differential equation

$$\partial\Phi/\partial t = \{ \lambda(t)\, s - \mu(t) \}\{s-1\}\, \partial\Phi/\partial s \tag{1.1}$$

with initial condition $\Phi(s; 0,0) = s$. The auxiliary equations are

$$dt / -1 = ds / \{ \lambda(t)\, s - \mu(t) \}\{s -1 \} = d\Phi / 0\, . \tag{1.2}$$

In the non-trivial case $\lambda(t) \neq c\mu(t)$, where c is some positive constant, Kendall (1948) noted that

$$ds / dt = - \lambda(t)\, s^2 + \{ \lambda(t) + \mu(t) \}s - \mu(t) \tag{1.3}$$

was a Riccati equation for which a general solution was available. Writing $s = 1 + w^{-1}$, Kendall effectively found

$$dw / dt = \{ \lambda(t) - \mu(t) \}w + \lambda(t)$$

with the solution

$$w\, e^{-\rho(t)} - \int_{(0,t)} \lambda(v)\, e^{-\rho(v)}\, dv = c_1$$

or

$$(s - 1)^{-1} e^{-\rho(t)} - \int_{(0,t)} \lambda(v)\, e^{-\rho(v)}\, dv = c_1\, , \tag{1.4}$$

where $\rho(t) = \int_{(0,t)} \{\lambda(u) - \mu(u)\}\, du$, $\rho(v) = \int_{(0,v)}\{\lambda(u) - \mu(u)\}\, du$ and c_1 is an arbitrary constant.

The p.g.f. of the process is thus given by

$$\Phi(s; 0, t) = f((s - 1)^{-1} e^{-\rho(t)} - \int_{(0,t)} \lambda(v)\, e^{-\rho(v)} dv)\, ,$$

where $f(\cdot)$ is an arbitrary function such that $\Phi(s; 0, 0) = f((s - 1)^{-1}) = s$. It is readily found that

$$\Phi(s; 0, t) = \{ A(t) + (1 - A(t) - B(t))\, s \} / \{1 - sB(t) \}\, , \tag{1.5}$$

where

$$B(t) = \int_{(0,t)} \lambda(v)\, e^{-\rho(v)}\, dv\} \{e^{-\rho(t)} + \int_{(0,t)} \lambda(v)\, e^{-\rho(v)}\, dv \}^{-1},$$

and

$$A(t) = \{ e^{-\rho(t)} + \int_{(0,t)} \lambda(v)\, e^{-\rho(v)}\, dv - 1\} \{e^{-\rho(t)} + \int_{(0,t)} \lambda(v)\, e^{-\rho(v)}\, dv\}^{-1}.$$

From (1.5), one obtains

$$P_{10}(t) = A(t), \quad P_{1n}(t) = (1 - P_{10}(t))(1 - B(t))\, B(t)^{n-1} \qquad n \geq 1, \tag{1.6}$$

and

$$E(X(t)) = e^{\rho(t)}, \quad Var(X(t)) = e^{2\rho(t)} \int_{(0,t)} e^{-\rho(v)} \{\lambda(v) + \mu(v)\} dv.$$

The probability of extinction $P_{10}(t)$ when $\lambda(t) \neq \mu(t)$ will tend to 1 as $t \to \infty$ when $\lambda(t) < \mu(t)$. For one can readily see that

$$e^{-\rho(t)} - 1 = \int_{(0,t)} \{\mu(v) - \lambda(v)\}\, e^{-\rho(v)}\, dv,$$

so that

$$P_{10}(t) = A(t) = \{ \int_{(0,t)} \mu(v)\, e^{-\rho(v)}\, dv \} / \{ 1 + \int_{(0,t)} \mu(v)\, e^{-\rho(v)}\, dv\},$$

which tends to 1 as $t \to \infty$, provided $\int_{(0,\infty)} \mu(v)\, e^{-\rho(v)}\, dv$ diverges as it does when $\lambda(t) < \mu(t)$. A similar result holds when $\lambda(t) = \mu(t)$, but when $\lambda(t) > \mu(t)$,

$$\lim_{t \to \infty} P_{10}(t) = \{ \int_{(0,\infty)} \mu(v)\, e^{-\rho(v)}\, dv\} / \{ 1 + \int_{(0,\infty)} \mu(v)\, e^{-\rho(v)}\, dv \} < 1,$$

where $\int_{(0,\infty)} \mu(v)\, e^{-\rho(v)}\, dv$ converges. Further details of the process may be found in Kendall (1948).

2. A BIVARIATE PARITY DEPENDENT PROCESS.

Ten years ago, Gani and Saunders (1976) considered a population of cells following a linear birth-death process, in which $\{ X_k(t) \}_{k=0,1,\ldots,\infty}$ denoted the numbers of cells of parity k=0,1,...,n,... respectively, where $X_k(t)$ cells had given rise to k offspring. Here, if the birth and death rates are λ, μ (constants), the p.g.f. $\Phi(s_0, s_1, \ldots; t) = E(\prod_{k=0,\ldots,\infty} s_k^{X_k(t)})$ of the process satisfies the partial differential equation

$$\partial\Phi / \partial t = \Sigma_k(\lambda\, s_0 s_{k+1} - (\lambda + \mu)\, s_k + \mu)\ \partial\Phi / \partial s_k \quad k=0,1,...,\infty , \tag{2.1}$$

with initial condition $\Phi(s_0,s_1,...;0)= s_0$, for simplicity.

The equation above has so far proved intractable, but some simplification of the model is possible. For example, if $R_1(t) = \Sigma_{k\geq 1} X_k(t)$ is the number of cells which have given rise to one or more offspring, and we write

$$G(s_0, s_1; t) = \Phi(s_0, s_1,...s_1; t)= E(s_0^{X_0(t)}\, s_1^{R_1(t)}),$$

it is readily seen that this satisfies the partial differential equation

$$\partial G/\partial t = \{ \lambda s_0 s_1 - (\lambda + \mu)s_0 + \mu \}(\partial G / \partial s_0)+ \{ \lambda s_0 s_1 - (\lambda + \mu)s_1 + \mu\}(\partial G / \partial s_1), \tag{2.2}$$

with initial condition $G(s_0, s_1; 0)= s_0$.

The auxiliary equations are

$$- dt / 1 = ds_0 / \{ \lambda s_0 s_1 - (\lambda + \mu)s_0 + \mu\} = ds_1 / \{ \lambda s_0 s_1 - (\lambda + \mu)s_1 + \mu\} = dG/0, \tag{2.3}$$

so that

$$- ds_0 / dt = \lambda s_0 s_1 - (\lambda + \mu)s_0 +\mu, \qquad - ds_1 / dt = \lambda s_0 s_1 - (\lambda + \mu)s_1 +\mu .$$

It is readily seen that

$$(s_0 - s_1)\, e^{- (\lambda+\mu)t} = c_1 \quad \text{or} \quad s_0 = s_1 + c_1 e^{(\lambda+\mu)t},$$

where c_1 is an arbitrary constant, so that

$$- ds_1 / dt = \lambda s_1^2 + \{ \lambda c_1 e^{(\lambda+\mu)t} - (\lambda+\mu)\}s_1 + \mu, \tag{2.4}$$

which is once again a Riccati equation.

In their recent paper, Srinivasan and Ranganathan (1983) reduce this, by a series of transformations to a modified Bessel's equation. Setting $x= - c_1 e^{(\lambda+\mu)t}$, (2.4) becomes

$$ds_1 / dx = - \lambda (\lambda+\mu)^{-1} s_1^2 x^{-1} + \{ (\lambda+\mu)^{-1} + x^{-1} \}s_1 - \mu (\lambda+\mu)^{-1} x^{-1} , \tag{2.5}$$

and the usal transformation $s_1= (\lambda + \mu)\, x\, \lambda^{-1} z^{-1}(dz / dx)$ for the Riccati equation now leads to the second order linear equation

$$(d^2z / dx^2) - (\lambda+\mu)^{-1} (dz / dx) + \mu(\lambda+\mu)^{-2}\, z\, x^{-2} = 0. \quad (2.6)$$

The further subtitution $z= y\, x^{1/2} \exp(x/2(\lambda+\mu))$ followed by $w= x/2(\lambda+\mu)$ transform (2.6) to the modified Bessel equation

$$w^2 (d^2y / dw^2) + w (dy / dw) - (w^2 + m^2)\, y = 0\,, \quad (2.7)$$

with $m^2= ((\lambda - \mu) / 2(\lambda + \mu))^2$.

The solution of (2.7) is known to be

$$y = AI_m(w) + BK_m(w)$$

where $I_m(w)$ and $K_m(w)$ are modified Bessel functions of the first and second kinds respectively. Srinivasan and Ranganathan (1983) show, on retracing the substitutions above, that

$$\{[(2\lambda s_1- (\lambda+\mu))x^{-1} - 1]K_m(w)-K'_m(w)\}\ \{I'_m(w)-[(2\lambda s_1- (\lambda+\mu))x^{-1} -1]I_m(w)\}^{-1}= c_2,$$

where c_2 is an arbitrary constant.

The general solution for $G(s_0, s_1; t)$ is given by

$$G(s_0, s_1; t)= f(c_1,c_2)\,,$$

where $f(\cdot,\cdot)$ is an an arbitrary function such that $G(s_0, s_1; 0)= s_0$. Using standard arguments, Srinivasan and Ranganathan finally arrive at the result

$$G(s_0,s_1;t) = (\lambda+\mu)(2\lambda)^{-1} + 2^{-1} (s_0 - s_1)e^{-(\lambda+\mu)t} \{1- H(\eta,- 2^{-1}\lambda (s_0 - s_1)e^{-(\lambda+\mu)t} \} \quad (2.8)$$

where

$$\eta=\{[(\lambda+\mu)(1- 2\theta)K_m(\theta)- \lambda(s_0 - s_1)K'_m(\theta)]\} / \lambda(s_0 - s_1)I'_m(\theta)- [(\lambda+\mu)(1- 2\theta)I_m(\theta)],$$

$$\theta = \lambda(s_1+s_0) / 2(\lambda + \mu)\,, \qquad H(\alpha, \beta) = \{ \alpha I'_m(\beta) + K'_m(\beta) \} / \{ \alpha I_m(\beta) + K_m(\beta) \}.$$

3. THE TWO-SEX POPULATION MODEL.

Goodman (1953) in an early *Biometrics* paper, discussed the two-sex birth-death process consisting of a populaion X(t) of females, and Y(t) of males in which the females alone give birth to females with rate λp and males with rate λq $(p+q=1)$. The death rates for females and males are μ and μ' respectively. Goodman obtained some moments of the process, and Srinivasan and Ranganathan (1983) were able to extend his results, also drawing attention to the relationship of this model with the parity dependent one.

An early attempt by Tapaswi and Roychoudhury (1983) to solve the problem fully was not entirely successful, but their more recent paper (1985) has provided a basic solution in all its analytic complexity. Gani and Tin (1986) obtained the same result simultaneously, though concentrating somewhat more on the structure of the process.

It is easy to show that if

$$\Phi(s_1, s_2; t) = \Sigma_{x,y} P_{x,y}(t)\, s_1^x s_2^y = E(s_1^{X(t)} s_2^{Y(t)}), \quad x,y = 0,\ldots,\infty,$$

is the p.g.f. of the process, then

$$\partial\Phi / \partial t = \{\lambda p s_1^2 + \lambda q s_1 s_2 - (\lambda+\mu)s_1 + \mu\}(\partial\Phi / \partial s_1) + \mu'(1- s_2)(\partial\Phi / \partial s_2) \qquad (3.1)$$

with the initial condition $\Phi(s_1, s_2; 0)= s_1$, for simplicity. The auxiliary equations are

$$dt/-1 = ds_1/\{\lambda p s_1^2 + \lambda q s_1 s_2 - (\lambda+\mu)s_1 + \mu\} = ds_2/\{\mu'(1- s_2)\} = d\Phi / 0.$$

From these, we obtain

$$(1 - s_2)\, e^{-\mu' t} = c_1,$$

where c_1 is an arbitrary constant, and

$$ds_1 / ds_2 = \lambda p\, s_1^2 / \{\mu'(1- s_2)\} + \{\lambda q s_2 - \lambda - \mu\}\, s_1 / \{\mu'(1- s_2)\} + \mu / \{\mu'(1- s_2)\} \quad (3.2)$$

which is again a Riccati equation. While Tapaswi and Roychoudhury (1985) reduce this to a confluent hypergeometric equation, we outline here the equivalent but more direct approach of Gani and Tin (1986).

Writing $s_1 = \mu'(s_2 - 1) / \lambda pz(dz / ds_2)$ and $s_2 = 1 - \theta$, we obtain

$$\theta^2 (d^2z / d\theta^2) + (\theta / \mu') (\mu' - \mu - \lambda p - \lambda q\theta) (dz / d\theta) + (\lambda\mu p/\mu'^2) z = 0 \qquad (3.3)$$

This is solved by using the standard method of the Frobenius power series

$$z(\theta) = \Sigma_n A_n \theta^{n+m} \quad , n=0,1,...,\infty , A_0 \neq 0$$

There are two solutions $z_1(\theta)$ and $z_2(\theta)$ of (3.3) corresponding to the roots $m_1 = \lambda p / \mu'$ and $m_2 = \mu / \mu'$ of the indicial equation. We finally obtain for the case $\lambda p \neq \mu$,

$$\Phi(s_1, s_2; t) = (\lambda p)^{-1} \{[H_1(u,\mu',v) H_2(\mu,\mu',r) - H_1(\lambda p,\mu',r) H_2(b,\mu',v) e^{-(\lambda p-\mu) t}] [H_1(u,\mu',v) H_2(1,0,r) - H_1(1,0,r) H_2(b,\mu',v) e^{-(\lambda p-\mu) t}]^{-1}\} \qquad (3.4)$$

where $u = \lambda p(s_1 - 1)$, $v = 1 - s_2$, $r = (1 - s_2) e^{-\mu' t}$, $b = \lambda ps_1 - \mu$, and

$$H_1(w_1,\mu' w_2,\theta)\theta^{\lambda p/\mu'} = (w_1 - \lambda pw_2)z_1(\theta) + \mu'\theta w_2 z'_1(\theta) ,$$

$$H_2(w_1,\mu' w_2,\theta)\theta^{\mu/\mu'} = (w_1 - \lambda pw_2)z_2(\theta) + \mu'\theta w_2 z'_2(\theta) .$$

A similar, but slightly more complex result holds when $\lambda p = \mu$.

We note that when $t \to \infty$, the probability of extinction is 1 if $\lambda p \leq \mu$ and $\mu / \lambda p$ if $\lambda p > \mu$ as one would expect.

4. CONCLUDING REMARKS.

The Riccati equation has the general form

$$ds / dt = P(t) s^2 + Q(t) s + R(t) \qquad (4.1)$$

and may be transformed into a linear differential equation of the second order by the substitution

$$s = - (P(t)z)^{-1} (dz / dt).$$

This leads to

$$P(\, d^2z / dt^2) - (\, PQ + (dP / dt)\,)\, (\, dz / dt\,) + P^2 Rz \; = \; 0\,, \tag{4.2}$$

and the above procedure was used to obtain the solutions of Section 2 and 3.

It is also true that if one solution $s = s_1(t)$ of the differential equation (4.1) is known, then the transformation $s = s_1(t) + w^{-1}$ will allow (4.1) to be integrated directly. In Section 1, $s_1(t) = 1$ was a solution of (1.3), and the transformation $s = 1 + w^{-1}$ led to the complete solution of the first order differential equation.

It is interesting to note that the structure of birth-death type processes is such that they will often give rise to a Riccati equation of the form (4.1), for which a useful solution can be found.

REFERENCES

GANI, J. AND SAUNDERS, I.W. (1976). On the parity of individuals in a branching process. *J. Appl. Prob.* **13**, 219-230.

GANI, J. AND TIN, P.(1986). A note on the two-sex population processs. *J. Appl. Prob.* **23**, A, 335-344.

GOODMAN, L. A. (1953). Population growth of sexes. *Biometrics* **9**, 212- 225.

KENDALL, D.G. (1948). On the generalized birth-and-death process. *Ann. Math. Statist.* **19**, 1-15

SRINIVASAN, S.K. AND RANGANATHAN, C.R. (1983). Parity dependent population growth models . *J. Math. Phy. Sci.* **17**, 279-292.

TAPASWI, P.K. AND ROYCHOUDHURY, R.K. (1983). A new approach to the distribution problems arising in studies of population growth of sexes. *Technical Report,* ISI Calcutta.

TAPASWI, P.K. AND ROYCHOUDHURY,R.K.(1985). A solution to the distribution problems arising in the studies of a two-sex population process. *Stoch. Proc. Applns* **19**, 359-370.