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ESTIMATION OF THE SPECTRAL MOMENT, BY MEANS OF THE EXTREMA

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An estimator of the standard deviation of the first derivative of a stationary Gaussian process with known variance and two continuous derivatives, based on the values of the relative maxima and minima , is proposed, and some of its properties are considered.

Key words: Estimation; Gaussian processes; Rice formula; Spectral moment; Stationary processes.

AMS Classification (1980): Primary, 60G10 ; Secondary: 60G15, 62F99.

Estimación del Momento Espectral Basado en Valores Extremos

Se propone un estimador de la desviación típica de la derivada de un proceso gaussiano estacionario de variancia conocida, basado en los valores de los extremos relativos del proceso, y se estudian algunas de sus propiedades.

Palabras clave: Estimación; Fórmula de Rice; Momento espectral; Prcesos estacionarios; Procesos gaussianos.

Clasificación AMS (1980): Primaria, 60G10 ; Secundaria, 60G15, 62F99.

1. INTRODUCTION

The mean frecuency γ of a centered stationary Gaussian process $\{X(t),\ t \in \mathbf{R}\}$ is defined by $\gamma^2 = \lambda_2 / \lambda_0$, where $\lambda_i = \int_{(-\infty,\infty)} \lambda^i \, dF(\lambda)$, i=0,2, and dF is the spectral measure related with the covariance

$$\Gamma(t) = E\,X(0)\,X(t) \quad \text{by} \quad \Gamma(t) = \int_{(-\infty,\infty)} e^{i\lambda t}\, dF(\lambda).$$

We shall assume that X has a.s. a continuous second derivative $X^{(2)}$. The variance $\lambda_0 = \Gamma(0) = \text{Var } X(t)$ will be assumed to be known, and, with no loss of generality, equal to one, hence

$$\gamma^2 = \lambda_2 = -\ \Gamma''(0) = E\ (X^{(1)}(t))^2 = \text{Var } X^{(1)}(t).$$

In order to estimate γ^2 or γ from the observation of X(t) during $0 \le t \le T$, two sorts of statistics have been proposed. The first one is due to S. Rice (1945) who introduced the unbiased estimator of γ^2

$$\gamma_i^2(T) = T^{-1} \int_{[0,T]} X^{(1)2}(t)dt \qquad (1)$$

with variance

$$\sigma_i^2(T) = (4/T) \int_{[0,T]} (1-t/T)\ (\Gamma''(t))^2\, dt \qquad (2)$$

satisfying

$$\lim_{T\to\infty} T\,\sigma_i^2(T) = 4 \int_{[0,\infty)} (\Gamma''(t))^2 dt. \qquad (3)$$

The other one is based on the use of the number $n_u^+(T)$ of upcrossings of X(t) through the level u, with expectation given by Rice formula

$$E\{ n_u^+(T) \} = T\,\gamma\,(2\pi)^{-1} \exp(-u^2/2) \qquad (4)$$

(the definition of upcrossings and a proof of (4) can be seen in Cramer and Leadbetter (1967), for instance). Steinberg et al. (1955), have proposed the use of the unbiased estimator $\gamma_{c,0}(T)$ of the parameter γ, where

$$\gamma_{c,u}(T) = 2\pi\ \Gamma^{-1} \exp(u^2/2)\ n_u^+(T), \quad u \in \mathbf{R} \qquad (5)$$

The properties of (1) and (5), and their relative efficiencies for different shapes of covariance functions, have been studied by G. Lindgren (1974). He also proposed an inprovement of $\gamma_{c,0}(T)$, namely, to select a finite number of values of u , say $u_1, u_2, \ldots, u_p$, and to combine linearly the corresponding $\gamma_{c,ui}(T)$ (i = 1,2,,..., p) to obtain

$$\gamma_c(T,a) = \Sigma_i\ a_i\,\gamma_{c,ui}(T) \qquad i = 1, \ldots ,p \qquad (6)$$

with the vector of weights $a = (a_1, \ldots, a_p)$ such that $\Sigma_i a_i = 1$, $i = 1, \ldots ,p$. He develops in particular the case $p = 3$, $u_1 = -u$, $u_2 = 0$, $u_3 = u$, and provides criteria to choose the weight vector.

As the finite value of p increases, the computation of the variance of $\gamma_c(T, a)$ becomes more and more involved, and the choosing of an adequate vector weights (minimizing the variance of $\gamma_c(T, a)$) requires too much computation work.

The purpose of the present paper is to show how a simple step forward, following Lindgren's ideas, leads to a new family of estimators, with variances easier to compute. Let us combine the infinite number of statistics (5) with a weight measure $d\alpha$ such that $\int_{(-\infty,\infty)} d\alpha(u) = 1$, in the form

$$\int_{(-\infty,\infty)} \gamma_{c,u}(T)\, d\alpha(u) \tag{7}$$

and notice that $n_u^+(T)$; and hence $\gamma_{c,u}(T)$, are sectionally constant as a function of *u*. In fact, the jumps of $n.^+(T)$ are a.s.

$$n_{u+0}^+(T) - n_{u-0}^+(T) = \{ -1 \text{ for } u \in \boldsymbol{M}, 1 \text{ for } u \in \boldsymbol{m} \} \tag{8}$$

with $\boldsymbol{M} = \{u : \text{for some } t \in (0, T], X \text{ has a maximum on } t \text{ and } X(t) = u\}$,
$\boldsymbol{m} = \{u : \text{for some } t \in [0, T), X \text{ has a minimum on } t \text{ and } X(t) = u\}$,
as it is esay to see, taking into account that the probability that two extrema have the same value is zero.

We introduce now the function

$$G(u) = \int_{(0,u)} 2\pi \exp(u^2/2)\, d\alpha(u) \tag{9}$$

and integrate (7) by parts:

$$\int_{(-\infty,\infty)} \gamma_{c,u}(T)\, d\alpha(u) = T^{-1}\int_{(-\infty,\infty)} n_u^+(T)\, dG(u)$$

$$= -T^{-1}\int_{(-\infty,\infty)} G(u)\, dn_u^+(T) = T^{-1}\left(\Sigma_{u\in \boldsymbol{M}} G(u) - \Sigma_{u \in \boldsymbol{m}} G(u) \right).$$

As a final step, we simplify our estimator, by neglecting the effect of the endpoints of (0, T), thus having what we shall call γ_e (e from "extrema"):

$$\gamma_e(T) = \gamma_e(T, G) = T^{-1}\left(\Sigma_{u\in \boldsymbol{M}} G(u) - \Sigma_{u\in \boldsymbol{m}} G(u) \right) \tag{10}$$

with

$$M = \{ X(t): \ X^{(1)}(t) = 0,\ X^{(2)}(t) < 0,\ t \in (0,T) \},$$
$$m = \{ X(t): \ X^{(1)}(t) = 0,\ X^{(2)}(t) > 0,\ t \in (0,T) \} \qquad (11)$$

(Notice that M and m are essentially $\boldsymbol{M} \cap (0, T) = \boldsymbol{M} \mid \{T\}$ and $\boldsymbol{m} \cap (0, T) = \boldsymbol{m} \mid \{0\}$).

The condition $\int_{(-\infty,\infty)} d\alpha(u) = 1$ imposed to the weight measure implies that the resulting estimator (7) is unbiased, and, since neglecting what happens at the endpoints of (0, T) should be irrelevant for large values of T , one would expect $\gamma_e(T)$ to be at least assymptotically unbiased as T goes to infinity. In fact, it turns to be unbiased for every T , as we shall see in the next section.

When our γ_e estimators are based on step functions G, they are almost equivalent to γ_c estimators. In general the γ_c are less efficient than γ_i, except for the case of a very concentrated spectrum of Γ, and the same should be expected to happen with γ_e . The advantage of either γ_c or γ_e is eventually their ease of computation.

2. EXPECTATION AND VARIANCE OF $\gamma_e(T)$.

The computation of the first and second moments of $\gamma_e(T)$ has been developped by S. Benzaquen (1983), with arguments similar to the ones used in Cramér and Leadbetter (1967) to derive Rice formula (4), or in Benzaquen and Cabaña (1982) to obtain the expected measure of level sets in the case of d-dimensional parameter processes. M. Wschebor (1982, 1983 a,b) has generalized those results and weakened their assumptions in a series of papers, and his methods could be applied in our present context to relax the assumptions. We limit ourselves to state Benzaquen's results with a suitable notation in view of our purposes, and to add some comments of heuristical nature to justify them, in Section 5.

With adequate assumptions on Γ, the estimator $\gamma_e(T)$ constructed from any bounded increasing function G satisfies:

$$E\{\gamma_e(T)\} = \gamma\,(2\pi)^{-1} \int_{(-\infty,\infty)} G(u)\ u\ \exp(-u^2/2)\, du$$
$$= -(2\pi)^{-1/2}\ E(\,G(X(0))\,X^{(2)}(0)\,) = \gamma \qquad (12)$$

and

$$E\{ \gamma_e(T) \} = 2\, T^{-2} \int_{[0,T]} \{ (T - t)\ \varphi_{X^{(1)}(0),\, X^{(1)}(t)} (0,0)$$

$$E(G(X(0))\, G(X(t))\, X^{(2)}(0)\, X^{(2)}(t) \mid X^{(1)}(0) = X^{(1)}(t) = 0) \} dt +$$

$$\gamma\, T^{-1} \int_{(-\infty,\infty)} G^2(u)\ (2\pi)^{-1}\ \exp(- u^2 / 2)\, \psi_\upsilon(u)\, du \qquad (13)$$

where $\varphi_{X^{(1)}(0),\, X^{(1)}(t)}$ is the joint density of $X^{(1)}(0)$, $X^{(1)}(t)$ and

$$\gamma^2 \psi_\upsilon(u) = E(|X^{(2)}(0)| \mid X(0) = u) = \gamma^2\, E\, |u + \upsilon Z| \qquad (14)$$

with z standard normal and $\upsilon^2 \gamma^4 = \Gamma^{iv}(0) = E\, X^{(2)2}$,

Benzaquen's assumptions are that Γ has four derivatives, the spectrum of $-\Gamma''$ has a continuous component, $\Gamma^{iv}(0) < \infty$ and $\int_{[0,\alpha]} t^{-1} (\Gamma^{iv}(0) - \Gamma^{iv}(t))\, dt < \infty$ for some positive constant α.

For a proof of (12) and (13), we refer to Benzaquen (1983). We end this section noticing that, since $\gamma = \varphi_{X^{(1)}(t)}\, E(-X^{(2)}(t)\, G(X(t)) | X^{(1)}(t) = 0)$ for every t, and $2T^{-2} \int_{[0,T]} (T - t)\, dt = 1$, then Var $\gamma_e(T)$ appears as the sum of the first integral in (13) minus $\gamma^2 = (E\gamma_e)^2$, namely

$$2\, T^{-2} \int_{[0,T]} (T-t)\, [\ \varphi_{X^{(1)}(0),\, X^{(1)}(t)} (0,0)$$

$$E(X^{(2)}(0)\, X^{(2)}(t)\, G(X(0))\, G(X(t)) \mid X^{(1)}(0) = X^{(1)}(t) = 0)$$

$$-\ \varphi_{X^{(1)}(0)}\ E(- X^{(2)}(0)\, G(X(0)) \mid X^{(1)}(0) = 0)\ \varphi_{X^{(1)}(t)}$$

$$E(- X^{(2)}(t)\, G(X(t)) \mid X^{(2)}(t) = 0)]\ dt \qquad (15)$$

plus the remaning integral

$$\gamma (2\pi T)^{-1} \int_{(-\infty,\infty)} G^2(u)\, \psi_\upsilon(u)\ \exp(- u^2/2)\, du \qquad (16)$$

3. CHOOSING THE WEIGHT FUNCTION G.

If the covariance decays for large t, the contributions of (15) to the variance of $\gamma_e(T)$ is expected to be lees important than the remaining term (16), because the bracket vanishes for large t and is bounded even for small t, in spite of the degeneracy of the joint density $\varphi_{X^{(1)}(0),X^{(1)}(t)}$, because the conditional expectation of $X^{(2)}(0)X^{(2)}(t)$ compensates this fact. Therefore, a reasonable goal to have in mind when selecting the function G is to minimize (16), with the restrictions

$$(2\pi)^{-1}\int_{(-\infty,\infty)} \exp(-u^2/2)\, dG(u) =$$
$$(2\pi)^{-1}\int_{(-\infty,\infty)} u\, G(u)\exp(-u^2/2)\, du = 1 \qquad (17)$$

and $\lim_{u\to\pm\infty} G(u)\exp(-u^2/2) = 0.$

A variational argument leads to the minimizing function

$$G(u) = k\,u \,/\, \psi_\upsilon(u) \qquad (18)$$

with the constant k chosen in such a way that (17) holds (See Table I below).

From (14) we have $\psi_\upsilon(u) = \upsilon\psi_1(u/\upsilon)$, and a straightforward calculation shows that ψ_1 is an odd function, and for $u > 0$, $\psi_1(u) = u - 2u\,\Phi(-u) + 2\varphi(u)$, with $\varphi(u) = (2\pi)^{-1/2}\exp(-u^2/2)$ and $\Phi(u) = \int_{(-\infty,u)} \varphi(z)\, dz$.

Substituting this expression in (18), we get $G(u) = kg(u/|\upsilon|)$, with

$$g(u) = \mathrm{sgn}(u)\,/\,(1-2\Phi(-|u|) + 2\varphi(u)/|u|).$$

For such G, (16) reduces to $\gamma k/t$, and we conjecture that this value provides a first approximation of Var γ_e. Furthermore, since for small t, $X^{(2)}(0)$ and $X^{(2)}(t)$ are negatively correlated, given $X^{(1)}(0) = X^{(1)}(t) = 0$, one should expect the actual value of Var γ_e to be less than $\gamma k/T$.

TABLE I. Values of $k = 2\pi(\int_{(-\infty,\infty)} u^2 \psi_\upsilon^{-1}(u) \exp(- u^2/2)\, du)^{-1}$ as a function of $\upsilon = \gamma^{-2} (EX^{(2)2})^{1/2}$

υ	k	υ	k
.2	3.1722	1.6	4.5888
.4	3.2579	1.8	4.8884
.6	3.3950	2.0	5.2002
.8	3.5760	2.5	6.0211
1.0	3.7912	3.0	6.8855
1.2	4.0364	3.5	7.7799
1.4	4.3038	4.0	8.6956

4. COMPARISON WITH RICE INTEGRAL ESTIMATOR AND NUMERICAL TEST OF THE CONJECTURE IN § 3, FOR PARTICULAR COVARIANCES

In order to compare the performances of our estimator and of γ_i, we supply in Table II cetain indicators corrresponding to some covariance functions, all of which have $\Gamma(0) = -\Gamma''(0) = 1$, namely

$$\Gamma(t) = \exp(-at^2/2) \cos(bt) \qquad (19)$$

for suitably chosen a and b. The indicators are:

(i) the values of the parameters a and b defining the covariance function,
(ii) the value of $k = k(\upsilon)$ (see (18)) which gives a first approximation of $V_e = \lim_{T\to\infty} T \operatorname{Var} \gamma_e(T)$,
(iii) the value of V_e , available from a numerical computation of (15), to test the conjecture that (15) is less important than (16), and negative, and
(iv) the value of $V_i = \int_{(-\infty,\infty)} (\Gamma''(t))^2 dt = \lim_{T\to\infty} T \sigma_i^2 (T)/4$, that is approximately equal to $\lim_{T\to\infty} T \operatorname{Var} \gamma_i (T)$.

The results in Table II validate the conjecture only for covariances with slow oscillations (b <<1) and strongly decaying for t >> 0 (a >> 0).

TABLE II. Comparison of k, V_e and V_i for covariances given by (19) ($\Gamma(0) = -\Gamma''(0) = \gamma^2 = 1$).

a	b	k	V_e	V_i
0.500	0.00	4.78	2.13	0.665
0.495	0.10	4.78	2.13	0.665
0.480	0.20	4.78	2.13	0.665
0.455	0.30	4.78	2.11	0.668
0.420	0.40	4.78	2.09	0.675
0.375	0.50	4.78	2.03	0.690
0.320	0.60	4.77	1.94	0.722
0.255	0.70	4.76	1.87	0.782
0.180	0.80	4.75	2.03	0.897
0.095	0.90	4.73	3.55	1.164
0.08595	0.91	4.72	3.92	1.213
0.07680	0.92	4.72	4.39	1.271
0.06755	0.93	4.72	4.95	1.341
0.04875	0.95	4.71	6.60	1.541

5. SOME HEURISTICS UNDERLYING THE COMPUTATION OF MOMENTS IN § 2.

When A = { $X^{(2)}$ vanishes a finite number of times during (0,T) and there is no $t \in (0,T)$ on which $X^{(1)}(t) = X^{(2)}(t) = 0$ } holds, the integrals

$$I_1(u,T) = \int_{(0,T)} \mathbf{1}_{\{X(t) < u\}} (- X^{(2)}(t)) \, G (X(t)) \, dt \tag{20}$$

and

$$I_2(u, v, (s',s''), (t', t'')) = \int_{(s',s'')} ds \int_{(t',t'')} \mathbf{1}_{\{X(1)(s) < u, \, X(1)(t) < v\}} X(s) X(t) G(X(s)) G(X(t)) dt \tag{21}$$

have derivatives

$$(\partial I_1(u,T) / \partial u) |_{u=0} =$$

$$\lim_{\delta \downarrow 0} \delta^{-1} \int_{(0,T)} \mathbf{1}_{\{0 \le X(t) < \delta\}} (- X^{(2)}(t)) \, G(X(t)) dt = T \, \gamma_e(T)$$

and, if $(s', s'') \cap (t', t'') = \phi$,

$$(\partial^2 I_2(u,v,(s' , s''), (t' ,t'')) / \partial u\, \partial v) |_{u=v=0}=$$

$$(s''- s')\, \gamma_e ((s', s'')) \;\; (t''- t')\gamma_e (t', t'')) ,$$

as a local change of variables $t \to X(t)$ easily shows.

Taking expectations in (20) and then differentiating, it follows that

$$\partial E\, I_1(u,T) / \partial u\, |_{u=0} = \int_{(0,T)} \varphi_{X^{(1)}(t)}(0)\, E(- X^{(2)}(t)\, G(X(t))|X^{(1)}(t)= 0)dt$$

Where $\varphi_{X^{(1)}(t)}$ denotes the density of $X^{(1)}(t)$. If the derivative can be introduced under the expectation and A holds, (12) is obtained. The analogue, starting from (21), leads to

$$\partial^2\, EI_2(u, v,(s', s''), (t',t'')) / \partial u \partial v\, |_{u=v=0} =$$

$$\int_{(s',s'')} ds \int_{(t',t'')} \varphi_{X^{(1)}(s),X^{(1)}(t)}(0,0)\, E(X^{(2)}(s)\, X^{(2)}(t)\, G(X(s))\, G(X(t))\, |$$

$$X^{(1)}(s) = X^{(1)}(t) = 0)\; dt \qquad (22)$$

and, again, if the derivative can be taken inside the expectation and A holds, $E(s'', s')\gamma_e((s', s''))(t''- t'\, \gamma_e((t',t'')))$ is equal to (22). All those expressions are additive functions of the rectangle $(s', s'') \times (t', t'')$, so that we can conclude (excluding points in the diagonal because of the requirement $(s',s'')\cap(t',t'') = \phi$) :

$$T^2 E(\gamma_e(T) - \Sigma_{t\in M \cup m} G^2(t)) =$$

$$\int_{(0,T)}ds\int_{(0,T)}\varphi_{X^{(1)}(s),X^{(1)}(t)}(0,0)E(X^{(2)}(s)X^{(2)}(t)G(X(s))G(X(t))|X^{(1)}(s)= X^{(1)}(t)=0)dt=$$

$$2\int_{(0,T)}(T- t)\varphi_{X^{(1)}(0),X^{(1)}(t)}(0,0)E(X^{(2)}(0)X^{(2)}(t)G(X(0))G(X(t))|X^{(1)}(0)=X(t)=0)dt.$$

In order to have (13), it remains to compute the expectation of $\Sigma_{t\in M \cup m}G^2(t)$. This is accomplished by replacing in (20) $- X^{(2)}(t)$ by $| X^{(2)}(t) |$ and $G(X(t))$ by $G^2(X(t))$, thus obtaining

$$E\, \Sigma_{t\in M \cap m} G^2(T) = \int_{(0,T)}\varphi_{X^{(1)}(t)}(0)\;\; E(|X^{(2)}(t)|\, G^2(X(t))\, |\, X^{(1)}(t) = 0)\; dt$$

and hence

$$T^{-2} E \Sigma_{t \in M \cup m} G^2(t) = T^{-1} (2\pi)^{-(1/2)} \gamma^{-1} E(|X^{(2)}(0)| G^2 X(0)) =$$

$$T^{-1} (2\pi)^{-(1/2)} \gamma^{-1} \int_{(-\infty,\infty)} E(|X^{(2)}(0)| |X(0) = u) G^2(u) (2\pi)^{-(1/2)} \exp(- u^2/2) du.$$

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