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STATISTICAL CHOICE OF NON-SEPARATED ONE-PARAMETER MODELS

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Introduction

The purpose of this paper is to study the asymptotic choice between two models $\{F(x|\alpha),\ \alpha \in A \subseteq \mathbb{R}\}$ and $\{G(x|\beta),\ \beta \in B \subseteq \mathbb{R}\}$, A and B being intervals but such that for (α_0, β_0), and only for this pair, we have $F(x|\alpha_0) = G(x|\beta_0)$.

As examples considerer the models with distribution functions $F(x|\alpha) = x^\alpha$, $(0 < \alpha < +\infty)$ and $G(x|\beta) = 1 - (1 - x)^\beta$, $(0 < \beta < +\infty)$ on the support $[0, 1]$ which coincide for the uniform distribution $(\alpha_0 = 1, \beta_0 = 1)$ or the models with densities $f(x|\alpha) = \alpha x^{\alpha-1} e^{-x^\alpha}$ (Weibull distribution) $(0 < \alpha < +\infty)$ and $g(x|\beta) = \frac{1}{\Gamma(\beta)} x^{\beta-1} e^{-x}$ (gamma distribution) $(0 < \beta < +\infty)$, on the support $\mathbb{R}_+$, which coincide for the exponential distribution $(\alpha_0 = 1, \beta_0 = 1)$.

In the sequel, as it happens in the examples, we will suppose that exist densities $f(x|\alpha)$ and $g(x|\beta)$ with respect to Lebesgue measure —this can be extended to densities with respect to other measures, as the counting measure, with possible randomization. For notational simplicity we will use the index 0 to denote the restriction of functions at $\alpha = \alpha_0$, $\beta = \beta_0$ or $(\alpha, \beta) = (\alpha_0, \beta_0)$.

We will have to consider different alternatives as α_0 or β_0 can be border points of A or B or interior points.

We will only analyse the cases $\alpha \geqslant \alpha_0$, $\beta \geqslant \beta_0$ (analogous to one-sided tests) —the case $\alpha_0 \geqslant \alpha$, $\beta_0 \geqslant \beta$ being dealt with by symmetry— and (α_0, β_0) an interior point of $A \times B$ (analogous to two-sided tests); mixed situations can be dealt with in the same lines.

The technique used is a sequence of two tests: first to test if $(\alpha, \beta) = (\alpha_0, \beta_0)$, owing to the singular situation of this pair and, after its rejection, to test for F (with $\alpha \neq \alpha_0$) or for G (with $\beta \neq \beta_0$).

In one-sided tests the statistics $\bar{p}_n = \frac{1}{n}\Sigma_1^n p(x_i)$ and $\bar{q}_n = \frac{1}{n}\Sigma_1^n q(x_i)$, for the IID sample $(x_1, ..., x_n)$, are fundamental as well as for the asymptotics of two-sided, tests shere

$$p(x) = \left.\frac{\partial \log f(x|\alpha)}{\partial \alpha}\right|_0 \quad \text{and} \quad q(x) = \left.\frac{\partial \log g(x|\beta)}{\partial \beta}\right|_0$$

A first example on a different, but similar, context can be seen in Tiago de Oliveira (1981) where the pre-history of the methodology presented here was develloped for two (or three) models which could be integrated in an over-all one-parameter model (θ) such that $\alpha, \beta \geqslant 0$ and $\theta = \alpha$ if the model was F and $\theta = -\beta$ if model was G; obviously $p(x) = -q(x)$. Note that the simpler methodology used in that paper can be used (locally) only if $p(x) = -q(x)$.

Note that if, instead of the parameters α and β, we had used $\alpha' = \alpha_0 + a(\alpha - \alpha_0)$ and $\beta' = \beta_0 + b(\beta - \beta_0)$, $a, b > 0$, the functions $p(x)$ and $q(x)$ would be substituted by $\frac{1}{a}p(x)$ and $\frac{1}{b}q(x)$. As the parameterization is not univoque, we will from the beginning «standardize» the parameters α and β supposing that the variances of $p(x)$ and $q(x)$ at 0 are equal to 1, otherways the equality of some derivatives would have no meaning at all, depending on the coefficient of the linear transformation used. This amounts, pratically, to substitute everywhere $p(x)$ and $q(x)$ by its standardized version. Remark that instead of imposing the equality of probabilities and derivatives we can substitute them by the condition that they will be in some (convenient) ratio.

As a final remark, we note that we will suppose that 1st and 2nd moments of p and q do exist always.

Statistical choice for the one-sided case

Let us denote by $\mathscr{L}_F(x|\alpha)$, $\mathscr{L}_G(x|\beta)$ and $\mathscr{L}_0(x)$ the likelihoods

$$\mathscr{L}_F(x|\alpha) = \prod_1^n f(x_i|\alpha)$$

$$\mathscr{L}_G(x|\beta) = \prod_1^n g(x_i|\beta)$$

and

$$\mathscr{L}_0(x) = \mathscr{L}_F(x|\alpha_0) = \mathscr{L}_G(x|\beta_0)$$

Let us denote by A_0, A_F and A_G the acceptance regions for $0(\alpha = \alpha_0, \beta = \beta_0)$, for $F(\alpha > \alpha_0)$ and for $G(\beta > \beta_0)$ and by $P_n(X|\xi)$ the probability of deciding $X(0$, F or $G)$ if the parameter is $\xi(0, \alpha$ or $\beta)$. The significance level of the decision will be $w_n = 1 - P_n(0|0) \to 0$. The probabilities for the first step of decision are (with a clear notation)

$$P_n(0|0) = \int_{A_0} \mathscr{L}_0(x)$$

$$P_n(0|\alpha) = \int_{A_0} \mathscr{L}(x|\alpha)$$

and

$$P_n(0|\beta) = \int_{A_0} \mathscr{L}_G(x|\beta)$$

where we are dealing with standardized parameters. If we impose

$$P_n(0|0) = 1 - w_n$$

$$\left.\frac{\partial P_n(0|\alpha)}{\partial \alpha}\right|_0 = \left.\frac{\partial P_n(0|\beta)}{\partial \beta}\right|_0 = \text{mín}$$

we obtain the conditions

$$\int_{A_0} \mathscr{L}_0(x) = 1 - w_n$$

$$\int_{A_0} \mathscr{L}_0(x)\left[\sum_1^n p(x_i) - \sum_1^n q(x_i)\right] = 0$$

$$\int_{A_0} \mathscr{L}_0(x)\sum_1^n p(x_i) = \text{mín}$$

which by the Neyman-Pearson theorem leads to the region

$$A_0{:}(1 - B_n')\sum_1^n p(x_i) + B_n'\sum_1^n q(x_i) \leqslant A_n'$$

where A_n' and B_n' are to be determined by the integral conditions above.

Once obtained A_0, let us search the regions A_F and A_G ($A_F \cup A_G = A_0^c$, $A_F \cap A_G = \phi$) from which we will decide for F (with $\alpha > \alpha_0$) and G (with $\beta > \beta_0$).

We have

$$P_n(F|\alpha) = \int_{A_F} \mathscr{L}_F(x|\alpha)$$

and

$$P_n(G|\beta) = \int_{A_G} \mathscr{L}_G(x|\beta)$$

Imposing

$$P_n(F|0) = P_n(G|0)$$

(i.e., splitting in half the significance level = misclassification error at 0) and the equal rates of deviation at 0 to be maximum we have

$$\left.\frac{\partial P_n(F|\alpha)}{d\alpha}\right|_0 = \left.\frac{\partial P_n(G|\beta)}{d\beta}\right|_0 = \max$$

we get the relations

$$\int_{A_F} \mathscr{L}_0(x) = \int_{A_G} \mathscr{L}_0(x)$$

$$\int_{A_F} \mathscr{L}_0(x)\sum_1^n p(x_i) = \int_{A_G} \mathscr{L}_0(x)\sum_1^n q(x_i) = \max$$

But as

$$\int_{A_G} \xi(x) = \int_{A_0^c} \xi(x) - \int_{A_F} \xi(x)$$

we get finally

$$\int_{A_F} \mathscr{L}_0(x) = w_n/2$$

$$\int_{A_F} \mathscr{L}_0(x)\left[\sum_1^n p(x_i) + \sum_1^n q(x_i)\right] = \int_{A_0^c} \mathscr{L}_0(x) \sum_1^n q(x_i) =$$

$$= -\int_{A_0} \mathscr{L}_0(x) \sum_1^n q(x_i)$$

$$\int_{A_F} \mathscr{L}_0(x) \sum_1^n p(x_i) = \max$$

By the Neyman-Pearson theorem we get

$$A_F : (1 - D_n') \sum_1^n p(x_i) - D_n' \sum_1^n q(x_i) \geqslant C_n'$$

restricted to the region A_0^c; A_G is defined by its complement in A_0^c. Recall, once more, that we are using standardized parameters in the formulation.

Let us now consider the asymptotic determination of A_n', B_n', C_n' and D_n'. Let $\mu_p(\alpha)$, $\sigma_p^2(\alpha)$, $\mu_q(\beta)$ and $\sigma_q^2(\beta)$ denote the mean values and variances of $p(x)$ and $q(x)$ with respect to F and G; with respect to $F(x|\alpha_0) = G(x|\beta_0)$ we know that $\mu_p(\alpha_0) = 0$, $\mu_q(\beta_0) = 0$. The use of standardized parameters amounts to substitute $p(x)$ by $q(x)/\sigma_p(\alpha_0)$ and $q(x)/\sigma_q(\beta_0)$ in the previous formulation when we intend to maintain the usual parameter as we will do. We will denote by ρ the correlation coefficient of $p(x)$ and $q(x)$ with respect to $F(x|\alpha_0) = G(x|\beta_0)$.

Then it is well know that, with $\bar{p}_n = \frac{1}{n}\sum_1^n p(x_i)$ and $\bar{q}_n = \frac{1}{n}\sum_1^n q(x_i)$, by the Central Limit Theorem we know that $\left(\sqrt{n}\,\frac{\bar{p}_n}{\sigma_p(\alpha_0)}, \sqrt{n}\,\frac{\bar{q}_n}{\sigma_q(\beta_0)}\right)$ is asymptoticaly binormal with standard margins and correlation coefficient ρ its density being denoted by $N''_\rho(.,.)$.

For the asymptotic determination of A'_n and B'_n let us, then, consider the auxiliary random variables

$$\xi_n = \frac{(1 - B'_n)\bar{p}_n}{\sigma_p(\alpha_0)} + \frac{B'_n\bar{q}_n}{\sigma_q(\beta_0)}$$

$$\eta_n = \frac{\bar{p}_n}{\sigma(\alpha_0)} - \frac{\bar{q}_n}{\sigma_q(\beta_0)}$$

The random pair (ξ_n, η_n), at 0, is asymptotically binormal with mean values zero, variances

$$V(\xi_n) = [(1 - B'_n)^2 + B'^2_n + 2\rho B'_n(1 - B'_n)]/n$$
$$V(\eta_n) = 2(1 - \rho)/n$$

and coefficient of correlation

$$\rho' = \frac{(1 - 2B'_n)\sqrt{1 - \rho}}{\sqrt{2}\sqrt{(1 - B'_n)^2 + B'^2_n + 2\rho B'_n(1 - B'_n)}}$$

using the asymptotic results as exact, the integral conditions

$$\int_{A_0} \mathscr{L}_0(x) = 1 - w_n$$

$$\int_{A_0} \mathscr{L}_0(x)\left\{\frac{\sum_1^n p(x_i)}{\sigma_p(\alpha_0)} - \frac{\sum_1^n q(x_i)}{\sigma_q(\beta_0)}\right\} = 0$$

can be written as

$$\int_{\xi_n \leqslant A'_n/n} \frac{1}{\sqrt{V(\xi_n)}\sqrt{V(\eta_n)}} N''_{\rho'}\left(\frac{\xi_n}{\sqrt{V(\xi_n)}}, \frac{\eta_n}{\sqrt{V(\eta_n)}}\right) = 0$$

$$\int_{\xi_n \leqslant A'_n/n} \frac{\eta_n}{\sqrt{V(\xi_n)}\sqrt{V(\eta_n)}} N''_{\rho'}\left(\frac{\xi_n}{\sqrt{V(\xi_n)}}, \frac{\eta_n}{\sqrt{V(\eta_n)}}\right) = 0$$

If $N(.)$ denotes the standard normal distribution function those equations can be written

$$N(A_n'/n\sqrt{V(\xi_n)} = 1 - w_n$$

$$\int_{\xi_n \leqslant A_n'/n} \int_{-\infty}^{+\infty} \eta_n N_\rho''\left(\frac{\xi_n}{\sqrt{V(\xi_n)}}, \frac{\eta_n}{\sqrt{V(\eta_n)}}\right) = 0$$

or

$$N(A_n'/n\sqrt{V(\xi_n)}) = 1 - w_n$$

$$\int_{\xi_n \leqslant A_n'/n} \rho' \frac{\xi_n}{\sqrt{V(\xi_n)}} N'\left(\frac{\xi_n}{\sqrt{V(\xi_n)}}\right) = 0$$

The 2nd equation leads to $\rho' = 0$ or equivalently $B_n' = 1/2$ as $\rho \neq 1$ and the 1st one to $A_n' = \sqrt{\dfrac{1+\rho}{2}}\sqrt{n}N^{-1}(1 - w_n)$. Thus A_0 is defined by

$$\sqrt{n}\left(\frac{\bar{p}_n}{\sigma_p(\alpha_0)} + \frac{\bar{q}_n}{\sigma_q(\beta_0)}\right) \leqslant \sqrt{2(1+\rho)}N^{-1}(1 - w_n) = A_n$$

Evidently, from the computational point of view, we can multiply by n, passing to sums and leading to

$$A_0 : \frac{1}{\sigma_p(\alpha_0)} \sum_1^n p(x_i) + \frac{1}{\sigma_q(\beta_0)} \sum_1^n q(x_i) \leqslant \sqrt{2(1+\rho)n}N^{-1}(1 - w_n)$$

As the conversion to sums is inmediate we will not return to it and maintain the use of averages which is more intuitive. Let us finally determine $A_F \subseteq A_0^c$.

We have the integral conditions

$$\int_{A_F} \mathscr{L}_0(x) = w_n/2$$

$$\int_{A_F} \mathscr{L}_0(x)\left(\frac{\sum_1^n p(x_i)}{\sigma_p(\alpha_0)} + \frac{\sum_1^n q(x_i)}{\sigma_q(\beta_0)}\right) = -\int_{A_0} \mathscr{L}_0(x)\frac{\sum_1^n q(x_i)}{\sigma_q(\beta_0)}$$

$$\int_{A_F} \mathscr{L}_0(x)\frac{\sum_1^n p(x_i)}{\sigma_p(\alpha_0)} = \max$$

so that

$$A_F:(1 - D_n')\frac{\sum_1^n p(x_i)}{\sigma_p(\alpha_0)} - D_n'\frac{\sum_1^n q(x_i)}{\sigma_q(\beta_0)} \geqslant C_n'$$

Let us show that if we take $D_n' = 1/2$ and $C_n' = 0$ we get a solution, when we use the asymptotic normal approximations. Let be $\xi_n' = \sqrt{n}\,\dfrac{\bar{p}_n}{\sigma_p(\alpha_0)}$ and $\eta_n' = \sqrt{n}\,\dfrac{\bar{q}_n}{\sigma_q(\beta_0)}$; the proposed region $A_F(\subseteq A_0^c)$ is $\xi_n' \geqslant \eta_n'$. Recall that A_0 is symmetrical in (ξ, η). We have

$$\int_{A_F} \mathscr{L}_0(x) \approx \int_{\{\xi' \geqslant \eta'\} \cap A_0^c} nN_\rho''(\xi', \eta')$$

$$\int_{\{\xi' \geqslant \eta'\} \cap A_0^c} nN_\rho''(\eta', \xi') = \int_{\{\xi' \geqslant \eta'\} \cap A_0^c} nN_\rho''(\xi', \eta') \simeq$$

$$\simeq \int_{A_G} \mathscr{L}_0(x) \quad \text{so that} \quad \int_{A_F} \mathscr{L}_0(x) \approx w_n/2$$

In the same way we can show that

$$\int_{A_F} \mathscr{L}_0(x)\frac{\sum_1^n p(x_i)}{\sigma_p(\alpha_0)} \approx \int_{A_G} \mathscr{L}_0(x)\frac{\sum_1^n q(x_i)}{\sigma_q(\alpha_0)}$$

Thus the result is

$$A_0: \frac{\bar{p}_n}{\sigma_p(\alpha_0)} + \frac{\bar{q}_n}{\sigma_q(\beta_0)} \leqslant \sqrt{\frac{2(1+\rho)}{n}}\, N^{-1}(1-w_n)$$

$$A_F: A_0^C \cap \left\{ \frac{\bar{p}_n}{\sigma_p(\alpha_0)} \geqslant \frac{\bar{q}_n}{\sigma_q(\beta_0)} \right\}$$

and

$$A_G: A_0^C \cap \left\{ \frac{\bar{p}_n}{\sigma_p(\alpha_0)} < \frac{\bar{q}_n}{\sigma_q(\beta_0)} \right\}$$

Let us now consider the example $F(x|\alpha) = x^\alpha (1 \leqslant \alpha \leqslant +\infty)$, $G(x|\beta) = 1 - (1-x)^\beta (1 \leqslant \beta \leqslant +\infty)$, $0 \leqslant x \leqslant 1$ which coincide for $\alpha_0 = \beta_0 = 1$ $(F(x|1) = G(x|1) = x)$. Then we get $p(x) = 1 + \log x$, $q(x) = 1 + \log(1-x)$, $\sigma_p^2(1) = \sigma_q^2(1) = 1$ (the parameters are naturally standard) and $\rho = 1 - \pi^2/6 = -0.6449341$.

We have thus:

$$A_0: 2n + \sum_1^n \log[x_i(1-x_i)] \leqslant 2\sqrt{1-\pi^2/12}\,\sqrt{n}N^{-1}(1-w_n)$$

$$A_F: A_0^C \cap \left\{ \sum_1^n \log x_i \geqslant \sum_1^n \log(1-x_i) \right\}$$

and

$$A_G: A_0^C \cap \left\{ \sum_1^n \log x_i < \sum_1^n \log(1-x_i) \right\}$$

Note that if $q = -p$ we have $\rho' = \pm($ (if $B_n \neq 1/2$) so that we can not conclude that $\rho' = 0$ because of the singularity of the binormal density.

Let us consider now this case. There we have only one family of distribution functions $\{M(x|\theta),\ \theta \in A \subseteq \mathbb{R}\}$ where the interior point $\theta_0 \in A$ is such that for $\theta > \theta_0$ we have one type of models and for $\theta < \theta_0$ another*. This is the case for the distribution of maxima under von Mises-Jenkinson formula

$$M(x|\theta) = e^{-(1+\theta x)^{-1/\theta}} \quad ; \quad \theta \in \mathbb{R}$$

* It may be useful to recall that to imbed two alternative models coinciding for α_0 and β_0 it is only possible if $-p(x) = q(x)$; as seen. But if this happens the general model $M(x|\theta) = F(x|\alpha_0 + \theta)$ if $\theta > 0$, $M(x|\theta) = G(x, \beta_0 - \theta)$ if $\theta \leqslant 0$ is a convenient form of local imbedding.

which for $\theta = 0^+ = 0^-$ leads to Gumbel model, for $\theta > 0$ to Fréchet model and for $\theta < 0$ to Weibull model. For more details and complements see Tiago de Oliveira (1981).

The technique of analysis used before could follow closely the cited paper. We sketch the proof in the lines above, now.

Let $\mathscr{L}(x|\theta)$ be the likelihood $[\mathscr{L}_0(x) = \mathscr{L}(x|0)]$ and suppose we want to decide in the trillemma $\theta < 0$, $\theta = 0$ or $\theta > 0$. Denoting by $P_n(X|\theta)$ the probability of deciding X, which way be $(-, 0, +)$, for the value θ of the parameter, we have then

$$P_n(0|\theta) = \int_{A_0} \mathscr{L}(x|\theta)$$

and $P_n(A_+|\theta) = \int_{A_+} \mathscr{L}(x|\theta)$ for the probability of deciding $\theta > 0$ if the parameter has the value θ.

Then the conditions are

$$P_n(A_+|0) = w_n/2$$

and

$$\left.\frac{dP_n(A_+|\theta)}{d\theta}\right|_0 = \max$$

which by the Neyman-Pearson leads to

$$\sum_1^n p(x_i) \geqslant A_n$$

or equivalently $\sqrt{n}\bar{p}_n > \bar{A}_n$. $\bar{A}_n$ is evidently given by the $1 - N(\bar{A}_n) = w_n/2$.

Evidently A_-, implying the decision that $\theta < 0$, is given by $\sqrt{n}\bar{p}_n < \bar{B}_n = -\bar{A}_n$.

Statistical choice for the two-sided case

Let us, now, suppose that (α_0, β_0) is an interior point of $A \times B \subseteq \mathbb{R}^2$. The conditions are, evidently,

$$P_n(0|0) = 1 - w_n$$

$$\left.\frac{dP_n(0|\alpha)}{d\alpha}\right|_0 = 0$$

$$\left.\frac{dP_n(0|\beta)}{d\beta}\right|_0 = 0$$

$$\left.\frac{d^2P_n(0|\alpha)}{d\alpha^2}\right|_0 = \left.\frac{d^2P_n(0|\beta)}{d\beta^2}\right|_0 = \min\ (<0)$$

or, equivalently,

$$\int_{A_0} \mathscr{L}_0(x) = 1 - w_n$$

$$\int_{A_0} \mathscr{L}_0(x) \sum_1^n p(x_i) = 0$$

$$\int_{A_0} \mathscr{L}_0(x) \sum_1^n q(x_i) = 0$$

$$\int_{A_0} \mathscr{L}_0(x) \left[\sum_1^n p_2(x_i) + \left(\sum_1^n p(x_i) \right)^2 - \sum_1^n q_2(x_i) - \left(\sum_1^n q(x_i) \right)^2 \right] = 0$$

$$\int_{A_0} \mathscr{L}_0(x) \left(\sum_1^n p_2(x_i) + \left(\sum_1^n p(x_i) \right)^2 \right) = \min\ (<0)$$

where, as before, $p_2(x) = \left.\frac{\partial^2 \log f(x|\alpha)}{\partial \alpha^2}\right|_0$ and $q_2(x) = \left.\frac{\partial^2 \log g(x|\beta)}{\partial \beta^2}\right|_0$.

The Neyman-Pearson theorem leads to (recall that we suppose for simplicity $\sigma_p(\alpha_0) = \sigma_q(\beta_0) = 1$).

$$\sum_1^n p_2(x_i) + (n\bar{p}_n)^2 \leqslant A_n' \left\{ \sum_1^n p_2(x_i) + (n\bar{p}_n)^2 - \sum_1^n q_2(x_i) - (n\bar{q}_n)^2 \right\}$$
$$+ B_n' n\bar{p}_n + C_n' n\bar{q}_n + D_n'$$

Dividing by n^2 and supposing the mean values of $p_2(x)$ and $q_2(x)$ to exist, by Khintchine theorem, $\frac{1}{n}\sum_1^n p_2(x_i)$ and $\frac{1}{n}\sum_1^n q_2(x_i)$ converge to them and we get, for large values of n, the region

$$A_0 : (1 - A_n')\overline{p}_n^2 + A_n'\overline{q}_n^2 \leqslant D_n$$

where, by symmetry, we take $A_n' = 1/2$. As we supposed $\sigma_p(\alpha_0) = \sigma_q(\beta_0) = 1$, the region, in general, is defined by

$$A_0 : \frac{\overline{p}_n^2}{\sigma_p^2(\alpha_0)} + \frac{\overline{q}_n^2}{\sigma_q(\beta_0)^2} \leqslant 2D_n \quad \text{or} \quad n\left(\frac{\overline{p}_n^2}{\sigma_p^2(\alpha_0)} + \frac{\overline{q}_n^2}{\sigma_q^2(\beta_0)}\right) \leqslant 2nD_n' = D_n$$

Introducing, once more, the standard variables

$$\xi_n = \sqrt{n}\,\frac{\overline{p}_n}{\sigma_p(\alpha_0)} \quad \text{and} \quad \eta_n = \sqrt{n}\,\frac{\overline{q}_n}{\sigma_q(\beta_0)}$$

the region can be written

$$\tilde{A} : \xi_n^2 + \eta_n^2 \leqslant D_n$$

and, by symmetry, the conditions

$$\int_{\tilde{A}_0} \xi N_\rho''(\xi, \eta) = 0$$

$$\int_{\tilde{A}_0} \eta N_\rho''(\xi, \eta) = 0$$

$$\int_{A_0} (\xi^2 - \eta^2) N_\rho''(\xi, \eta) = 0$$

approximations to the corresponding conditions for the exact A_0, are used.

The value of D_n is computed through

$$\int_{\tilde{A}_0} N_\rho''(\xi, \eta) = 1 - w_n$$

Let us, now, determine A_F and $A_G(A_F \cup A_G = A_0^C, A_F \cap A_G = \phi)$. A_F and A_G may be defined by

$$P_n(F|0) = P_n(G|0) = w_n/2$$

$$\left.\frac{dP_n(F|\alpha)}{d\alpha}\right|_0 = \left.\frac{dP_n(G|\beta)}{d\beta}\right|_0 = 0$$

$$\left.\frac{d^2P_n(F|\alpha)}{d\alpha^2}\right|_0 = \left.\frac{d^2P_n(G|\beta)}{d\beta^2}\right|_0 = \max$$

Using the Neyman-Pearson theorem and the asymptotic approximation, in the same lines as before, we get

$$\tilde{A}_F : \left\{(1 - A_n')\frac{\bar{p}_n^2}{\sigma_p(\alpha_0)^2} - A_n'\frac{\bar{q}_n^2}{\sigma_q(\beta_0)^2} \geqslant D_n'\right\} \cap A_0^C$$

Let us show, also, that $A_n' = 1/2$, $D_n' = 0$ is a solution, that is

$$\tilde{A}_F : A_0^C \bigcap \left\{\frac{|\bar{p}_n|}{\sigma_p(\alpha_0)} \geqslant \frac{|\bar{q}_n|}{\sigma_q(\beta_0)}\right\}$$

We have

$$\int_{\tilde{A}_F} N_\rho''(\xi, \eta) = \int_{(\xi^2 \geqslant \eta^2) \cap A_0^C} N_\rho''(\xi, \eta) = \int_{(\eta^2 \geqslant \xi^2) \cap A_0^C} N_\rho''(\xi, \eta) = \int_{\tilde{A}_G} N_\rho''(\xi, \eta)$$

so that as

$$P_n(\tilde{A}_F|0) = P(\tilde{A}_G|0) \quad \text{and} \quad P_n(\tilde{A}_F|0) + P(\tilde{A}_G|0) \approx P(A_0^C|0) = w_n$$

we get $P_n(\tilde{A}_F|0) \approx w_n/2$ and $P_n(\tilde{A}_G|0) \approx w_n/2$.

Also $\int_{\tilde{A}_F} \xi N_\rho''(\xi, \eta) = 0 = \int_{\tilde{A}_G} \eta N_\rho''(\xi, \eta)$ by symmetry and

$$\int_{\tilde{A}_F} (\xi^2 + \eta^2) N_\rho''(\xi, \eta) - \int_{A_0^C} \xi^2 N_\rho''(\xi, \eta) =$$

$$= \int_{\tilde{A}_F} \eta^2 N_\rho''(\xi, \eta) - \int_{\tilde{A}_G} \xi^2 N_\rho''(\xi, \eta) \to 0$$

and the remnant of the equation converges to zero.

Evidently $\tilde{A}_G$ is defined by

$$\tilde{A}_G : A_0^C \bigcap \left\{ \frac{|\bar{p}_n|}{\sigma_p(\alpha_0)} < \frac{|\bar{q}_n|}{\sigma_q(\beta_0)} \right\}$$

For the example we have

$$A_0 : n\left\{ \left(1 + \frac{1}{n}\sum_1^n \log x_i\right)^2 + \left(1 + \frac{1}{n}\sum_1^n \log(1 - x_i)\right)^2 \right\} \leqslant D_n$$

$$\tilde{A}_F : A_0^C \bigcap \left\{ \left|1 + \frac{1}{n}\sum_1^n \log x_i\right| \geqslant \left|1 + \frac{1}{n}\sum_1^n \log(1 - x_i)\right| \right\}$$

$$\tilde{A}_G : A_0^C \bigcap \left\{ \left|1 + \frac{1}{n}\sum_1^n \log x_i\right| < \left|1 + \frac{1}{n}\sum_1^n \log(1 - x_i)\right| \right\}$$

In case we wanted to deal with the statistical choice betwen the Weibull and gamma distributions, with densities:

$$f(x|\alpha) = \alpha x^{\alpha - 1} e^{-x^\alpha} \quad \text{and} \quad g(x|\beta) = \frac{1}{\Gamma(\beta)} x^{\beta - 1} e^{-x} \quad ; \quad x, \alpha, \beta > 0$$

which coincide in the exponential for $\alpha = 1$ and $\beta = 1$ we have

$$p(x) = 1 + \log x - x \log x$$
$$q(x) = \gamma + \log x$$
$$\sigma_p^2(1) = \pi^2/6 + (1 - \gamma)^2$$
$$\sigma_q^2(1) = \pi^2/6$$

and

$$\rho = \frac{\gamma}{\sqrt{\frac{\pi^2}{6}} \times \sqrt{\frac{\pi^2}{6} + (1 - \gamma)^2}}$$

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