TRABAJOS DE ESTADISTICA Y
DE INVESTIGACION OPERATIVA
Vol. 36. Núm. 1, 1985, pp. 129 a 140

NOTAS

CONCOMITANTS AND LINEAR ESTIMATORS IN AN i-DIMENSIONAL EXTREMAL MODEL

M. Ivette Gomes
Departamento de Estatística,
Investigação Operacional e Computação
Centro de Estatística e Aplicações
da Universidade de Lisboa

Abstract

We consider here a multivariate sample $\underset{\sim}{X}_j = (\underset{\sim}{X}_{1,j} > \cdots > X_{i,j})$, $1 \leqq j \leqq n$, where the $\underset{\sim}{X}_j$, $1 \leqq j \leqq n$, are independent i-dimensional extremal vectors with suitable unknown location and scale parameters λ and δ respectively. Being interested in linear estimation of these parameters, we consider the multivariate sample $\underset{\sim}{Z}_j$, $1 \leqq j \leqq n$, of the order statistics of largest values and their concomitants, and the best linear unbiased estimators of λ and δ based on such multivariate sample. Computational problems associated to the evaluation of $\mu_i^{(n)}$ and $\Sigma_i^{(n)}$, the mean value and the covariance matrix of standardized $\underset{\sim}{Z}_j$, $1 \leqq j \leqq n$, are also discussed.

1. Second order structure of the order statistics and their concomitants in an *i*-dimensional extremal model

1.1. PROBABILISTIC SET-UP

Let $\{X_n^*\}_{n \geqq 1}$ be a stationary sequence of random variables (r.v.'s) and let $\underset{\sim}{M}_n$ be the i-th dimensional random vector $(M_n^{(1)}, ..., M_n^{(i)})$, where $M_n^{(j)}$ is the j-th largest order statistic (o.s.) of $(X_1^*, ..., X_n^*)$, $1 \leqq j \leqq i$, $n \geqq i$, i a fixed integer.

Under certain weak conditions (Gomes, 1978, 1979) it is possible to show that if there exist normalizing constants $\{a_n\}_{n\geqq 1}$ ($a_n > 0$) and $\{b_n\}_{n\geqq 1}$ and a non-degenerate distribution function (d.f.) $G(x)$ such that

$$\lim_{n\to\infty} P[M_n^{(1)} \leqq a_n x + b_n] = G(x)$$

for all x in the set of continuity points of $G(.)$, then

$$\lim_{n\to\infty} P\left[\bigcap_{j=1}^{i} M_n^{(j)} \leqq a_n x_j + b_n\right] =$$

$$= G\left(\min_{1\leqq j\leqq i} x_j\right) \sum_{r_{j-1}\leqq r_j\leqq j-1} \cdots \sum \prod_{j=1}^{i-1} \log \frac{G\left(\min\limits_{1\leqq k\leqq j} x_k\right)}{G\left(\min\limits_{1\leqq k\leqq j+1} x_k\right)}^{r_{j+1}-r_j} \Big/ (r_{j+1} - r_j)!, \quad (r_1 = 0) \quad (1.1)$$

where, according to Gnedenko's theorem, $G(x) = \exp\left(-(1 + \theta x)^{-1/\theta}\right)$, $x \in \mathbb{R}$, $1 + \theta x > 0$ (if $\theta = 0$, we get $G(x) = \exp\left(-\exp(-x)\right)$, $x \in \mathbb{R}$, Gumbel's d.f.).

A random vector of dimension i with joint d.f. given by the right hand-side of (1.1) is called an *i-dimensional extremal vector.*

1.2. STATISTICAL ASPECTS

The results previously mentioned become important in many practical applications involving extremes. Indeed, one of the drawbacks pointed out to Gumbel's approach to statistical inference using extrema is the wasting of information by only considering the maxima of groups of observations, though generally records of the top few o.s. are available. It is then of interest to develop inference techniques for dealing with multivariate samples $(\underset{\sim}{X}_1, \ldots, \underset{\sim}{X}_n)$ of independent random vectors with d.f. given by the right-hand side of (1.1) and x_k replaced by $(x_k - \lambda)/\delta$, $1 \leqq k \leqq i$, where λ and δ are a location and a scale parameter respectively to be estimated from the sample (for a similar model, see Weissman (1978)).

1.3. LINEAR ESTIMATORS OF THE UNKNOWN PARAMETERS. CONCOMITANTS

Let us consider the multivariate sample $(\underset{\sim}{X}_1, \ldots, \underset{\sim}{X}_n)$ of indepent random vectors, where $\underset{\sim}{X}_j = (X_{1.j}, \ldots, X_{i.j})$ has a probability density function (p.d.f.)

$$f_{X_j}(x_1, \ldots, x_i; \lambda, \delta) = \exp\left(-\exp\left(-(x_i - \lambda)/\delta\right) - \sum_{k=1}^{i} (x_k - \lambda)/\delta\right)/\delta^i$$

$$x_1 > \cdots > x_i \quad ; \quad 1 \leqq j \leqq n$$

i a fixed integer. This p.d.f. has proved to be fruitful in the statistical analysis of extremes and corresponds to the joint d.f. given by the right-hand side of (1.1), $G(.)$ Gumbel's d.f.

In the univariate case the notion of o.s. plays an important role in statistical methods and is clear and unambiguous. For multivariate samples no reasonable basis exists for a full ordering of the data, but different generalizations of the concept of order can be made in two or more dimensions (V. Barnett, 1976, makes an interesting critical review of the subject).

We shall consider here the ordering of the largest values $(X_{1.1}, X_{1.2}, ..., X_{1.n})$. We then do not modify $\underset{\sim}{X}_r$, $1 \leqq r \leqq n$, we merely order them according to the ordering of the largest values (David (1973), David and Galambos (1974) consider the case $i = 2$ in a multivariate normal situation). We thus get the ordered sample $(\underset{\sim}{Z}_1^{(n)}, ..., \underset{\sim}{Z}_n^{(n)})$, where $\underset{\sim}{Z}_j^{(n)} = (Z_{1.j}^{(n)}, ..., Z_{i.j}^{(n)})$ and for every j, $1 \leqq j \leqq n$, there is an $m_j \in \{1, 2, ..., n\}$—the m_j's all different—such that $\underset{\sim}{Z}_j^{(n)} = \underset{\sim}{X}_{m_j}$. The $Z_{s.j}^{(n)}$, $2 \leqq s \leqq i$, are called the concomitants of the o.s. of largest values.

We were then interested in developing best linear unbiased estimators (BLUE) of the unknow parameters λ and δ based on o.s. of largest values and their concomitants and so we needed to derive the mean value $\mu_i^{(n)}$ and the covariance matrix $\Sigma_i^{(n)}$ of $\underset{\sim}{Y}_j^{(n)} = (\underset{\sim}{Z}_j^{(n)} - \lambda)/\delta$, $1 \leqq j \leqq n$, where λ denotes the column vector of dimension i with all its components equal to λ.

For the case $i = 1$ several problems arrived in the computation of the covariance matrix of the o.s. of Gumbel populations. The fact that the expression for $E(Y_{1.j}^{(n)})$ and $E(Y_{1.j}^{(n)} Y_{1.k}^{(n)})$, $1 \leqq j \leqq k \leqq n$ involve sums of terms large in magnitude and alternating in sign lead Lieblein (1954) to compute such covariance matrix only for sample sizes up to $n = 6$. Later on, with the aid of powerful computers and the use of recurrence relations on the mean values, Mann (1963) derived the coefficients of the BLUE of λ and δ for Gumbel populations and for sample sizes up to $n = 25$.

In our computation and in order to try to get higher precision and to reduce the number of independent calculations for the evaluation of $\mu_i^{(n)}$ and $\Sigma_i^{(n)}$ we have used not only direct formulas but also the following recurrence relations (assume for simplicity that $i = 2$)

$$E((Y_{r.j+1}^{(n)})^m) = (nE((Y_{r.j}^{(n-1)})^m) - (n-j)E((Y_{r.j}^{(n)})^m))/j \qquad n \geqq 2 \quad ; \quad 1 \leqq j \leqq n-1 \quad ; \quad r = 1, 2 \tag{1.2}$$

$$E(Y_{r.j+1}^{(n)} Y_{s.m+1}^{(n)}) = (nE(Y_{r.j}^{(n-1)} Y_{s.m}^{(n-1)}) - (m-j)E(Y_{r.j}^{(n)} Y_{s.m+1}^{(n)}) - (n-m)E(Y_{r.j}^{(n)} Y_{s.m}^{(n)}))/j$$

$$(r, s) \in \{1, 2\} X \{1, 2\} \quad ; \quad n \geqq 2 \quad ; \quad 1 \leqq j \quad , \quad m \leqq n-1 \quad \text{if} \quad r \neq s \quad \text{or} \quad 1 \leqq j < m \leqq n-1 \quad \text{if} \quad r = s \tag{1.3}$$

For the mean values $\mu_2^{(n)}$ there are no special problems on the accuracy of the results. The same happens for variances. However, even the initial values needed for the computation of the non-diagonal members of $\Sigma_2^{(n)}$ are cumbersome and usually involve the computation of sums of the type

$$\sum_{m=\alpha_1}^{\alpha_2} \binom{\beta_1}{m} \frac{(-1)^{\beta_1 - m}}{(mp + q)(\beta_2 - rm)} A(\beta_2 - m, m + s)$$

$$p \geqq 0 \quad ; \quad q \geqq 0 \ \text{ if } \ p \neq 0 \ \text{ or } \ q \geqq 1 \ \text{ if } \ p = 0 \quad ; \quad s \geqq 0 \quad ;$$

$$\beta_2 \geqq \beta_1 \geqq \alpha_2 \geqq \alpha_1 \geqq 0$$

where $A(j, k)$ is given below in 2.1.

In the computation of such covariances FORTRAN IV double precision was incorporated throughout the main program and associated sobroutines. The constants were read with the maximum possible machine precision and Spence's integral was computed with 22 correct decimal figures.

In the computations we have used two different algorithms:

Algorithm 1: Using direct formulas for mean values, variances and covariances.

Algorithm 2: Using the recurrence relations pointed out, with the respective initial conditions.

In the ICL1906 both methods were run, the first one being obviously much more expensive than the second one. For $i = 2$ the following checks were made:

$$\text{(a)} \quad \sum_{j=1}^{n} \sum_{m=1}^{n} \operatorname{Cov}(Y_{1\cdot j}^{(n)}, Y_{1\cdot m}^{(n)}) = n \operatorname{Var}\big((X_{1\cdot 1} - \lambda)/\delta\big) = n\psi'(1)$$

$$\text{(b)} \quad \sum_{j=1}^{n} \sum_{m=1}^{n} \operatorname{Cov}(Y_{2\cdot j}^{(n)}, Y_{2\cdot m}^{(n)}) = n \operatorname{Var}\big((X_{2\cdot 1} - \lambda)/\delta\big) = n\psi'(2) \qquad (1.4)$$

$$\text{(c)} \quad \sum_{j=1}^{n} \sum_{m=1}^{n} \operatorname{Cov}(Y_{1\cdot j}^{(n)}, Y_{2\cdot m}^{(n)}) = n \operatorname{Cov}\big((X_{1\cdot 1} - \lambda)/\delta, (X_{2\cdot 1} - \lambda)/\delta\big) = n\psi'(2)$$

where $\psi(.)$ is the digamma function, $\psi'(.)$ its derivative.

Using algorithm 1 the two sides of (a) agree up to 21 decimal figures for sample size $n = 2$, but the number of matching figures decreases rapidly as n increases, and for $n = 15$ the two sides agree up to 10 decimal figures. Using algorithm 2. and for $n = 15$ we still obtain an agreement of 19 decimal figures. Both algorithms are equivalent in respect of checks (b) and (c), the agreement being up to 9 decimal figures for $n \leqq 12$ and up to 8 decimal figures for $12 < n \leqq 20$. However, for $n = 15$ we get a negative determinat of the covariance matrix using both methods, which shows that the accuracy is not enough for further computations.

We then have run the second algorithm in the CDC 7600 (description of the program provided below). The agreement between the programs run in the two computers is the following: for $n < 10$ we have got an agreement of all eight decimal figures printed, for $n = 10$ we have an agreement of at least 7 decimal figures and for $n > 10$ the number of matching figures decreases rapidly as n increases, the agreement being of only 3 decimal figures for $n = 15$.

In the 7600 the two sides of (a) agree up to 27 decimal figures for $n \leqq 6$, 26 decimal figures for $6 < n \leqq 12$, 25 decimal figures for $12 < n \leqq 15$ and 24 decimal figures for $15 < n \leqq 20$. The two sides of either (b) or (c) agree up to 13 decimal figures for $n \leqq 12$ and up to 12 decimal figures for $12 < n \leqq 20$. Up to $n = 20$, the results obtained, for the covariance matrix of the o.s. of largest values only, agree with those given by White (1964). However, even in the CDC 7600 we receive a message of ill-conditioned matrix for sample size $n \geqq 18$.

2. Description of the program run in the CDC 7600

Let $\underset{\sim}{Y}^{(n)} = (Y_{1.1}^{(n)}, ..., Y_{1.n}^{(n)}, Y_{2.1}^{(n)}, ..., Y_{2.n}^{(n)})$ denote the standardized random vector of the o.s. of largest values and their concomitants in a 2-dimensional extremal model, $\mu_2^{(n)}$ the column vector of the mean values of $\underset{\sim}{Y}^{(n)}$ and $\Sigma_2^{(n)}$ the covariance matrix of $\underset{\sim}{Y}^{(n)}$.

2.1. External functions and subroutines

The functions needed in the evaluation of the initial values for $\mu_2^{(n)}$ and $\Sigma_2^{(n)}$ are, with γ denoting Euler's constant,

$$\theta_1(j) = (\gamma + \log j)/j$$

$$\theta_2(j) = (\pi^2/6 + (\gamma + \log j)^2)/j$$

$$H_2(j) = (\pi^2/6 - 1 + (\gamma + \log j - 1)^2)/j^2$$

$$A(j, k) = ((k - j)\theta_2(j + k) + (j\theta_1(j))^2 - 2L(1 + k/j) + \pi^2/6)/(2kj)$$

if $k < j$, with $A(j, k) + A(k, j) = \theta_1(j)\theta_1(k)$

$$A(j, j) = (\gamma + \log j)^2/(2j^2)$$

$$L(1 + x) = \int_1^{1+x} \{\log t/(t - 1)\} dt, \quad \text{Spence's integral,}$$

$\psi(j)$, the digamma function, and

$$B(j,k) = (kA(j,k) - \theta_1(j) + \theta_1(j+k) - kH_2(j+k))/k^2 \quad \text{if} \quad k \neq j$$

$$B(j,j) = (\gamma^2 - \pi^2/6 + 4\log 2 - 2\gamma\log 2 - (\log 2)^2 + (\log j)^2 - \\ - 2\log 2 \log j + 2\gamma \log j)/(4j^3)$$

The sobroutines involved provide

$$S(j_1, j_2, j_3, n_1) = \sum_{k=j_1}^{j_2} \binom{j_3}{k} (-1)^{j_3-k} A(n_1 - k, k+1)$$

$$S1(j_1, j_2, j_3, n_1, n_2) = \sum_{k=j_1}^{j_2} \binom{j_3}{k} (-1)^{j_3-k} A(n_1 - k, k+1)/(n_2 - k)$$

$$S2(j_1, j_2, j_3, n_1, n_2) = \sum_{k=j_1}^{j_2} \binom{j_3}{k} (-1)^{j_3-k} [A(n_1 - k, k+1) + A(1, n_1) - \\ - \gamma\theta_1(n_1 - k)]/(k(n_2 - k))$$

and

$$S3(j_1, j_2, j_3, n_1) = \sum_{k=j_1}^{j_2} \binom{j_3}{k} (-1)^{j_3-k} [A(n_1 - k, k+1) + \\ + A(1, n_1) - \gamma\theta_1(n_1 - k)]/k$$

2.2. Evaluation of $\mu_2^{(n)}$ and $\Sigma_2^{(n)}$

The development of this evaluation is made in the following steps:

Step 1: Computation of the functions $\theta_1(.)$, $\theta_2(.)$, $H_2(.)$, $A(.,.)$, $\psi(.)$ and $B(.,.)$, described in 2.1, for suitable values of the arguments.

Step 2: Computation of the initial mean values

$$E(Y_{1:1}^{(n)}) = n\theta_1(n), \quad n \geqq 1, \quad E((Y_{1:1}^{(n)})^2) = n\theta_2(n), \quad n \geqq 1$$

$$E(Y_{2:1}^{(n)}) = n(\gamma - \theta_1(n))/(n-1), \quad n > 1, \quad E((Y_{2:1}^{(n)})^2) = n(\pi^2/6 + \gamma^2 - \\ - \theta_2(n))/(n-1), \quad n > 1$$

$$E(Y_{1:1}^{(n)} Y_{2:1}^{(n)}) = nA(1, n-1), \quad n > 1$$

$$E(Y_{1:1}^{(n)} Y_{1:k}^{(n)}) = k(k-1)\binom{n}{k} S(0, k-2, k-2, n-1), \quad 1 < k \leqq n$$

and, with $D(j) = B(1,j) + B(j,1)$,

$$E(Y_{1\cdot k}^{(n)} Y_{2\cdot 1}^{(n)}) = k(k-1)\binom{n}{k}[(-1)^{k-2}D(n-1) - S3(1, k-2, k-2, n-1)]$$

$$\text{if} \quad 1 < k \leqq n$$

$$E(Y_{2\cdot 1}^{(n)} Y_{2\cdot k}^{(n)}) = \begin{cases} k(k-1)\binom{n}{k}[S2(1, k-2, k-2, n-1, n-2) - \\ \quad -(-1)^{k-2}D(n-1)/(n-2)] + n(n-1)(\gamma^2 - 2A(1, n-1))/ \\ \quad /((n-2)(n-k)) \quad \text{if} \quad 1 < k < n \\ -n(n-1)\gamma^2(\gamma + \psi(n-1))/(n-2) + \\ \quad + n(n-1)[S2(1, n-3, n-2, n-1, n-2) + \\ \quad + ((1-\gamma)\theta_1(n-1) - \gamma(1-\gamma) + \\ \quad + 2((\gamma + \psi(n-1))A(1, n-1) + D(n-1) - \\ \quad - (-1)^{n-2}D(n-1))/(n-2) + \\ \quad + (2A(1, n-1) - \gamma^2)/(n-2)^2] \quad \text{if} \quad k = n > 2 \\ (1-\gamma^2)/2 \quad \text{if} \quad k = n = 2 \end{cases}$$

$$E(Y_{1\cdot 1}^{(n)} Y_{2\cdot k}^{(n)}) = \begin{cases} n(n-1)A(1, n-1)/(n-k) - \\ \quad - k(k-1)\binom{n}{k} S1(0, k-2, k-2, n-1, n-2) \\ \quad \text{if} \quad 1 < k < n \\ n(n-1)[-(\gamma + \psi(n-1))A(1, n-1) - \\ \quad - S1(0, n-3, n-2, n-1, n-2) + \\ \quad + (\gamma - 1)\theta_1(n-1) - D(n-1)] \quad \text{if} \quad 1 < k = n \end{cases}$$

Step 3: Use of recurrence relations of type (1.2) and (1.3) to compute $\mu_2^{(n)}$ and the mean values $E(Y_{1\cdot j}^{(n)} Y_{1\cdot k}^{(n)})$, $1 \leqq j \leqq k \leqq n$, $E(Y_{1\cdot j}^{(n)} Y_{2\cdot k}^{(n)})$, $1 \leqq j, k \leqq$ and $E(Y_{2\cdot j}^{(n)} Y_{2\cdot k}^{(n)})$, $1 \leqq j \leqq k \leqq n$.

Step 4: Printing of the mean values computed in steps 2 and 3.

Step 5: Checks on the computations (see (1.4)).

Step 6: Computation and printing of $\Sigma_2^{(n)}$.

2.3. Computation of BLUE of unknown parameters

The computation of the BLUE of unknown parameters λ and δ, based on o.s. of largest values and concomitants, corresponding to the bivariate extremal sample $(\underset{\sim}{X}_1, ..., \underset{\sim}{X}_n)$ described in 1.3 is computationally straightforward. The BLUE of $\theta = \begin{bmatrix} \lambda \\ \delta \end{bmatrix}$ is

$$\theta^* = \begin{bmatrix} \lambda^* \\ \delta^* \end{bmatrix} = (P'_{2n}(\Sigma_2^{(n)})^{-1}P_{2n})^{-1}P'_{2n}(\Sigma_2^{(n)})^{-1} \begin{bmatrix} \underset{\sim}{Z}_1^{(n)} \\ \vdots \\ \underset{\sim}{Z}_n^{(n)} \end{bmatrix}, \quad \text{where}$$

$$P_{2n} = [\underset{\sim}{1}_{2n} | \mu_2^{(n)}] \quad ; \quad \underset{\sim}{1}_{2n}$$

being the column vector of size $2n$, with all its components equal to unity. Weights for obtaining such estimators are provided in table 1 below, for $n \leqq 12$ (for larger n, available from the author, on request).

Table 1. Weights for obtaining the BLUE of location parameter λ and scale parameter δ, based on o.s. of largest values and concomitants, and in a 2-dimensional extremal model.

$a_{n \cdot j}^{(1)}$—weight for $Z_{1 \cdot j}^{(n)}$ when estimating λ.
$a_{n \cdot j}^{(2)}$—weight for $Z_{2 \cdot j}^{(n)}$ when estimating λ.
$b_{n \cdot j}^{(1)}$—weight for $Z_{1 \cdot j}^{(n)}$ when estimating δ.
$b_{n \cdot j}^{(2)}$—weight for $Z_{2 \cdot j}^{(n)}$ when estimating δ.

n	j	$a_{n \cdot j}^{(1)}$	$a_{n \cdot j}^{(2)}$	$b_{n \cdot j}^{(1)}$	$b_{n \cdot j}^{(2)}$
2	1	.200379	.341839	.411444	−.051843
	2	.194052	.263730	.340438	−.700039
3	1	.125134	.232337	.226830	.028097
	2	.138753	.223471	.226830	−.166856
	3	.123661	.156644	.189158	−.558735
4	1	.090969	.174462	.153442	.040439
	2	.096369	.176169	.182904	−.051659
	3	.107754	.160549	.214087	−.193865
	4	.090078	.103649	.127891	−.473239
5	1	.071475	.139146	.114958	.040703
	2	.073889	.143108	.133306	−.012718
	3	.079746	.138720	.152803	−.086008
	4	.088555	.122037	.173270	−.197474
	5	.070613	.072711	.095714	−.414553

Table 1 (cont.)

n	j	$a_{n,j}^{(1)}$	$a_{n,j}^{(2)}$	$b_{n,j}^{(1)}$	$b_{n,j}^{(2)}$
6	1	.058876	.115476	.091523	.038259
	2	.059965	.119771	.104037	.003388
	3	.063355	.119048	.117196	−.041234
	4	.068406	.112586	.131319	−.101854
	5	.075417	.096327	.145915	−.193501
	6	.057970	.052802	.076136	−.371184
7	1	.050064	.098554	.075847	.035334
	2	.050495	.102657	.084926	.010759
	3	.052600	.103340	.094380	−.019252
	4	.055783	.100698	.104449	−.057440
	5	.060085	.093537	.115280	−.109111
	6	.065829	.078107	.126281	−.186977
	7	.049120	.039131	.063054	−.337531
8	1	.043554	.085878	.064664	.032535
	2	.043634	.089650	.071552	.014259
	3	.045001	.090903	.078667	−.007312
	4	.047136	.089964	.086175	−.033597
	5	.049988	.086456	.094212	−.066965
	6	.053687	.079144	.102857	−.112012
	7	.058506	.064617	.111481	−.179753
	8	.042588	.029295	.053729	−.310490
9	1	.038548	.076040	.056304	.030014
	2	.038433	.079466	.061709	.015872
	3	.039346	.080935	.067256	−.000889
	4	.040840	.080834	.073062	−.019596
	5	.042838	.079874	.079224	−.042979
	6	.045379	.075173	.085839	−.072619
	7	.048598	.067948	.092947	−.112572
	8	.052720	.054290	.099910	−.172561
	9	.037574	.021965	.046762	−.288184
10	1	.034577	.068189	.049826	.027788
	2	.034354	.071294	.054182	.016506
	3	.034972	.072819	.058628	.003802
	4	.036051	.073152	.063251	−.010855
	5	.037506	.072321	.068117	−.028172
	6	.039339	.070124	.073300	−.049238
	7	.041610	.066063	.078875	−.075914
	8	.044445	.059030	.084854	−.111828

Table 1 (cont.)

n	j	$a_{n.j}^{(1)}$	$a_{n.j}^{(2)}$	$b_{n.j}^{(1)}$	$b_{n.j}^{(2)}$
	9	.048026	.046172	.090606	−.165692
	10	.033606	.016351	.041367	−.269403
11	1	.031351	.061783	.044666	.025833
	2	.031067	.064600	.048251	.016613
	3	.031488	.066107	.051894	.006406
	4	.032284	.066670	.055662	−.005152
	5	.033375	.066364	.059601	−.018504
	6	.034746	.065112	.063763	−.034278
	7	.036421	.062680	.068207	−.053454
	8	.038463	.058582	.072993	− 077716
	9	.040987	.051784	.078115	−.110351
	10	.044137	.039654	.082955	−.159250
	11	.030389	.011955	.037071	−.253327
12	1	.028678	.056459	.040461	.024113
	2	.028361	.059024	.043465	.016429
	3	.028645	.060477	.046505	.008044
	4	.029244	.061159	.049635	−.001309
	5	.030080	.061164	.052889	−.011924
	6	.031133	.060472	.056303	−.024192
	7	.032410	.058970	.059921	−.038683
	8	0.33943	.056427	.063791	−.056289
	9	.035791	.052349	.067961	−.078548
	10	.038058	.045800	.072414	−.108464
	11	.040859	.034325	.076547	−.153258
	12	.027729	.008452	.033572	−.239380

In table 2, we present, up to the factor δ^2, the covariance matrix of BLUE based on o.s. ($i = 1$) and on o.s. and concomitants ($i = 2$). In the univariate situation we have obviously $\mu_1^{(n)} = H\mu_2^{(n)}$, $\Sigma_1^{(n)} = H\Sigma_2^{(n)}H'$, where $H = [I_n | 0_n]$, I_n the identity matrix of order n and 0_n the square matrix of order n with all its elements equal to zero. The BLUE of θ was then computed mainly in order to check the results with the ones previously obtained by other authors.

Table 2. Covariance matrix of BLUE based on o.s. (i = 1) and on o.s. and concomitants (i = 2)

$e_{11}^{(1)}$—Variance of BLUE of λ ($i = 1$)
$e_{22}^{(1)}$—Variance of BLUE of δ ($i = 1$)
$e_{12}^{(1)}$—Covariance of BLUE of λ and δ ($i = 1$)
$e_{11}^{(2)}$—Variance of BLUE of λ ($i = 2$)
$e_{22}^{(2)}$—Variance of BLUE of δ ($i = 2$)
$e_{12}^{(2)}$—Covariance of BLUE of λ and δ ($i = 2$)
$d_1 = e_{11}^{(1)}e_{22}^{(1)} - e_{12}^{(1)^2}$ $\quad d_2 = e_{11}^{(2)}e_{22}^{(2)} - e_{12}^{(2)^2}$

n	$e_{11}^{(1)}$	$e_{22}^{(1)}$	$e_{12}^{(1)}$	d_1	$e_{11}^{(2)}$	$e_{22}^{(2)}$	$e_{12}^{(2)}$	d_2
2	.6596	.7119	.0643	.4664	.4097	.3472	.1933	.1049
3	.4029	.3447	.0248	.1383	.2725	.2005	.1246	.0391
4	.2935	.2253	.0347	.0649	.2040	.1391	.0917	.0200
5	.2314	.1666	.0340	.0374	.1630	.1059	.0725	.0120
6	.1912	.1320	.0314	.0242	.1358	.0853	.0600	.0080
7	.1629	.1091	.0286	.0170	.1163	.0713	.0511	.0057
8	.1420	.0930	.0261	.0125	.1017	.0612	.0445	.0042
9	.1258	.0809	.0239	.0096	.0937	.0536	.0394	.0033
10	.1130	.0716	.0220	.0076	.0813	.0476	.0353	.0026
11	.1025	.0642	.0203	.0062	.0739	.0429	.0320	.0026
12	.0938	.0582	.0189	.0051	.0677	.0389	.0293	.0018
13	.0865	.0532	.0176	.0043	.0625	.0357	.0270	.0015
14	.0802	.0489	.0165	.0037	.0580	.0329	.0250	.0013
15	.0748	.0453	.0156	.0032	.0541	.0305	.0233	.0011

Acknowledgements: The author is grateful to INIC (Institute for Science Research, Lisbon) for financial support to present a preliminary version of this paper at COMPSTAT 80.

REFERENCES

BARNETT, V. (1976): «The ordering of multivariate data», *J. Royal Statist. Soc.* A139, 318-354.

DAVID, H. A. (1973): «Concomitants of order statistics». *Proc. 39th Session I.S.I.*, Vienna.

DAVID, H. A., and GALAMBOS, J. (1974): «The asymptotic theory of concomitants of order statistics», *J. Appl. Probab.* 11, 762-770.

GNEDENKO, B. V. (1943): «Sur la distribution limite du terme maximum d'une série aléatoire», *Ann. Math.* 44, 423-453.

GOMES, M. I. (1978): «Some Probabilistic and Statistical Problems in Extreme Value Theory», *Ph. D. Thesis, Univ. Sheffield.*

GOMES, M. I. (1979): «Extremal *i*-variate laws in stationary sequences», *Rev. Univ. Santander*, vol. 2, 1017-1019.

LIEBLEIN, J. (1954): «A new method of analizing extreme value data», *Nat'l Advisory Comm. for Aeronautics Techn. Note* 3053.

MANN, N. R. (1963): «Optimum estimates of parameters of continuous distributions», *Rocketdyne Research Report*, 63-41, California.

WEISSMAN, I. (1978): «Estimation of parameters and large quantiles based on the *k* largest observations», *J. Amer. Statist. Assoc.* 73, 812-815.

WHITE, J. S. (1964): «Least square unbiased linear estimation for the log-Weibull (extreme value) distribution», *I.E.E.E. Trans. Reliability* R-21, 89-93.