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SOME PROPERTIES OF BETA FUNCTIONS AND THE DISTRIBUTION FOR THE PRODUCT OF INDEPENDENT BETA RANDOM VARIABLES

Giorgio Pederzoli
Department of Quantitative Methods
Concordia University
Montreal, Quebec, Canada.

ABSTRACT

Products of independent beta random variables appear in a large number of problems in multivariate statistical analysis. In this article we show how a convenient factorial expansion of gamma ratios can be suitably used in deriving the exact density for a product of independent beta random variables. Possible applications of this result for obtaining the exact densities of the likelihood ratio criteria for testing hypotheses in the multinormal case are also pointed out. For the sake of illustration, the exact null density of Wilks'Λ for testing linear hypothesis in the real Gaussian case is derived. Furthermore, it will be shown that this method is applicable also to problems of a more general nature.

Key words and phrases: Gamma ratios, factorial expansions, product of independent beta variables, exact density, distributions of test statistics.

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1. Introduction

Products of independent real beta type-1 random variables appear in a variety of statistical problems. Several test statistics for testing hypotheses in the multinormal case can, for instance, be considered to be products of independent beta random variables. Several such cases are pointed out in Anderson (1958) and also in Mathai and Saxena (1973). The exact density for the product of independent beta random variables has been worked out by several authors by using different methods. A summary of the various techniques which are applicable in such context is available in Mathai (1973). Some of the commonly used methods involve the asymptotic expansions of gammas in the moment expressions and then obtain the density by inversion. In this article we will consider a different type of expansion for gamma ratios connected with the moments of beta random variables. This factorial type expansion is obtained with the help of ordinary binomial expansions, several of which are available in Erdelyi (1953).

2. Factorial Expansion of Gamma Ratios

In this section an expansion of factorial type for gamma ratios will be developed. The result will then be applied in order to derive the exact density for the product of independent real beta variables. We first consider a few preliminary lemmas.

Lemma 1. Let α and h be two complex numbers such that $R(\alpha) > 0$, $R(\alpha + h) > 0$ and β be a positive integer, where $R(\cdot)$ denotes the real part of $(\cdot)$. Then

$$(\beta - 1)!\prod_{j=1}^{\beta}(\alpha + h + \beta - j)^{-1} = \sum_{r=0}^{\beta-1}\binom{\beta - 1}{r}(-1)^r(\alpha + h + r)^{-1} \tag{2.1}$$

Proof. Let X be a random variable with beta type-1 density given by

$$f(x) = \frac{\Gamma(\alpha + \beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha - 1}(1 - x)^{\beta - 1},$$

$$0 < x < 1, \quad R(\alpha) > 0, \quad R(\beta) > 0 \tag{2.2}$$

The h-the moment of X is given by

$$E(X^h) = \frac{\Gamma(\alpha + h)}{\Gamma(\alpha)} \frac{\Gamma(\alpha + \beta)}{\Gamma(\alpha + \beta + h)} =$$

$$= \frac{\Gamma(\alpha + \beta)}{\Gamma(\alpha)} \frac{1}{(\alpha + h + \beta - 1)\ldots(\alpha + h)} \tag{2.3}$$

if β is a positive integer. However, it is also true that

$$E(X^h) = \frac{\Gamma(\alpha + \beta)}{\Gamma(\alpha)\Gamma(\beta)} \int_0^1 x^{\alpha+h-1}(1 - x)^{\beta-1} dx$$

$$= \frac{\Gamma(\alpha + \beta)}{\Gamma(\alpha)\Gamma(\beta)} \sum_{r=0}^{\beta-1} \binom{\beta - 1}{r} (-1)^r \int_0^1 x^{\alpha+h+r-1} dx \tag{2.4}$$

when β is a positive integer. Now, taking the integral, equating (2.3) with (2.4), and using the fact that $\Gamma(\beta) = (\beta - 1)!$ when β is a positive integer, yield the result.

Lemma 2. Let α, β, h be complex numbers such that $R(\alpha) > 0$, $R(\beta) > 0$, $R(\alpha + h) > 0$, where $R(\cdot)$ denotes the real part of $(\cdot)$. Then

$$\frac{\Gamma(\alpha + h)}{\Gamma(\alpha + \beta + h)} = \frac{1}{\Gamma(\beta)} \sum_{r=0}^{\infty} \frac{(1 - \beta)_r}{r!} (\alpha + h + r)^{-1} \tag{2.5}$$

where, for example, $(a)_m = a(a + 1)(a + 2)...(a + m - 1)$.

The proof of lemma 2 is the same as that of lemma 1. When β is a positive integer in fact, the infinite series on the right-hand side of (2.5) reduces to a finite sum and the result (2.5) becomes (2.1).

Lemma 3. Let α_i, $i = 1, ..., k$ be complex numbers such that $R(\alpha_i) > 0$ and r_i, $i = 1, ..., k$ be non-negative integers. Let h be a complex number such that $R(\alpha_i + h) > 0$, $i = 1, ..., k$. Let $\alpha_i + r_i \neq \alpha_j + r_j$ for all i and j, $i \neq j$. Then

$$\prod_{i=1}^{k} (\alpha_i + r_i + h)^{-1} = \sum_{i=1}^{k} \frac{a_i}{\alpha_i + r_i + h}, \text{ where} \tag{2.6}$$

$$a_j = \prod_{\substack{i=1 \\ i \neq j}} (\alpha_i - \alpha_j + r_i - r_j)^{-1} \tag{2.7}$$

The result follows trivially by using the simple partial fraction technique. It may be noted that for some values of α_i, α_j, r_i and r_j, if $\alpha_i + r_i =$ $= \alpha_j + r_j$ then one can use the generalized partial fraction technique developed in Mathai and Rathie (1971) and write the left-hand side of (2.6) as a sum. The coefficients a_j's however, will be more complicated and some of the factors will not be linear in h.

Theorem 1. Let $X_1, \ldots, X_k$ be k independent beta type-1 random variables with parameters (α_i, β_i), $i = 1, \ldots, k$ such $R(\alpha_i) > 0$, $R(\beta_i) > 0$, $i = 1, \ldots, k$ and $\alpha_l + r_i \neq \alpha_j + r_j$ for all i and j, $i \neq j$, where r_i, $i = 1, \ldots, k$ are non-negative integers. Let $Y = X_1 X_2 \ldots X_k$. Then the density of Y, denoted by $g(y)$ is given by,

$$(2.8)\quad g(y) = C \sum_{r_1=0}^{\infty} \ldots \sum_{r_k=0}^{\infty} \frac{(1-\beta_1)_{r_1}}{r_1!} \ldots \frac{(1-\beta_k)_{r_k}}{r_k!} \left\{ \sum_{j=1}^{k} a_j y^{\alpha_j + r_j - 1} \right\}$$

for $0 < y < 1$, where

$$(2.9)\qquad C = \prod_{i=1}^{k} \frac{\Gamma(\alpha_i + \beta_i)}{\Gamma(\alpha_i)\Gamma(\beta_i)}$$

The a_j's are given in (2.7) and the symbol $(a)_m$ is explained in (2.5).

Proof. Since the beta random variables $X_1, \ldots, X_k$ are independent, the h-th moment of Y is given by

$$(2.10)\qquad E(Y^h) = \prod_{j=1}^{k} E(X_j^h) = C' \prod_{j=1}^{k} \frac{\Gamma(\alpha_i + h)}{\Gamma(\alpha_j + \beta_j + h)}, \text{ where}$$

$$(2.11)\qquad C' = \prod_{j=1}^{k} \frac{\Gamma(\alpha_j + \beta_j)}{\Gamma(\alpha_j)}$$

Expanding the gamma ratios in $\Delta(h) = \prod_{j=1}^{k} \frac{\Gamma(\alpha_j + h)}{\Gamma(\alpha_j + \beta_j + h)}$ by lemma 2 and taking the product, one obtains

(2.12)
$$E(Y^h) = C \sum_{r_1=0}^{\infty} \cdots \sum_{r_k=0}^{\infty} \frac{(1-\beta_1)_{r_1} \cdots (1-\beta_k)_{r_k}}{r_1! \cdots r_k!} \left\{ \prod_{j=1}^{k} \frac{1}{(\alpha_j + r_j + h)} \right\}$$

where C is as defined in (2.9). Now, writing $\prod_{j=1}^{k} (\alpha_j + r_j + h)^{-1}$ as a sum by using lemma 3, and then taking the inverse Mellin transform, the result follows.

Corollary 1. If, in addition to the conditions of the theorem, $\beta_1, \ldots, \beta_k$ are positive integers, then

(2.13)
$$g(y) = C \sum_{r_1=0}^{\beta_1-1} \cdots \sum_{r_k=0}^{\beta_k-1} \binom{\beta_1-1}{r_1} \cdots \binom{\beta_k-1}{r_k} (-1)^{r_1+\cdots+r_k} \left\{ \sum_{j=1}^{k} a_j y^{\alpha_j+r_j-1} \right\}$$

The proof follows directly from lemma 1 and theorem 1.

It may be interesting to remark that if for some α_j and r_j any of the factors in $\Delta(h)$ are repeated, then one needs only the generalized partial fraction technique of Mathai and Rathie (1971) to put $\Delta(h)$ into a sum. In this case, when the inverse Mellin transform is taken, factors of the type $(-\log y)^m$ for $m = 0, 1, 2, \ldots$ will be brought in, but the coefficients a_j's of (2.7) will then be too complicated.

3. Applications to Test Statistics

The method developed in section 2 can be applied to a large number of test statistics in multivariate analysis. For many of these problems the h-th null moments of the likelihood ratio criteria or a one-to-one function of the likelihood criteria are equivalent to the h-th moment of a product of independent beta type-1 random variables. A discussion of the test statistics may be found in Anderson (1958), and their distributions are available in Mathai and Saxena (1973). The present technique is actually applicable to all such cases. For example, the h-th null moment of Wilks'Λ for testing linear hypothesis in the real Gaussian case has the following structure.

(3.1)
$$E(U^h) = C_p \prod_{j=1}^{p} \frac{\Gamma\left(\frac{n}{2} - \frac{j-1}{2} + h\right)}{\Gamma\left(\frac{n}{2} - \frac{j-1}{2} + \frac{q}{2} + h\right)}$$

where C_p is a normalizing constant such that $E(U^0) = 1$, $n \geqq p$, $q > 0$. This shows that U is structurally of the form

$$U = X_1 \cdots X_p \tag{3.2}$$

where $X_1, ..., X_p$ are independent real beta type-1 random variables with X_i having as parameters $\alpha_i = \frac{n}{2} - \frac{i-1}{2}$ and $\beta_i = \frac{q}{2}$ for $i = 1, ..., p$. If $p = 2$, then $\alpha_1 + r_1 \neq \alpha_2 + r_2$ for $r_1, r_2 = 0, 1, 2, ...$ and therefore the density of U is available from theorem 1 by putting $\alpha_1 = \frac{n}{2}$, $\alpha_2 = \frac{n}{2} - \frac{1}{2}$, $\beta_1 = \beta_2 = \frac{q}{2}$, $k = 2$ and $y = u$. If q is an even integer, then the result is available from corollary 1. If $p \geqq 3$, then it is evident that for some values of r_i and r_j, $\alpha_i + r_i = \alpha_j + r_j$. In this case there will be repeated factors in (2.6) and hence the density will contain terms of the type $u^{\alpha}(-\log u)^m$ with $\alpha > 0$ and $m = 0, 1, 2, ...$ As mentioned earlier, this case involves the application of the generalized partial fraction technique in order to go from (2.6) to (2.8). Although the expressions will lead to slightly complicated forms, the method is readily applicable, and, whenever the β_j's happen to be positive integers, one gets the density in the form of a finite sum. For the Wilks'Λ it is obvious that when $p = 2$ one can apply many other methods some of which are discussed in Anderson (1958) some more are available in Mathai and Saxena (1973).

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