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NOTAS

ESTIMATION OF PARAMETERS IN EXPONENTIAL AUTOREGRESSIVE MODELS

B. Chandra and M. K. Jena
Department of Mathematics
Indian Institute of Technology.
New Delhi.

ABSTRACT

The paper discusses the estimation of parameters in second order exponential autoregressive models.

1. INTRODUCTION

The traditional Box-Jenkins autoregressive moving average models (Box and Jenkins (1976)) are mainly suitable for modelling time series with marginal Gaussian distribution. In many practical problems we face with positive random variables. To tackle such type of time series Lawrance (1980) and Lawrance and Lewis (1980) developed models with exponential marginal distribution. In these models the independent innovation sequence is also exponential and the autocorrelation structure is determined by linear difference equations. The basic difference between these models and the Box-Jenkins ARMA models is that although both are linear in the variables and parameters, the linearity of these models is associated with the probabilistic choice between several linear combination of independent variables.

Lawrance and Lewis (1980) have introduced pth order autoregressive model in exponential variables (EAR(p) model) and linked this model with the exponential moving average model of qth order (EMA(q)) into a positively correlated EARMA (p, q) model. In this paper an attempt has been made for estimating the parameters in second order exponential autoregressive models.

2. EXPONENTIAL AUTOREGRESSIVE MODEL

The linear first autoregressive model for a stationary sequence of a random variable $[X_i]$ is defined as follows:

$$X_i = \alpha X_{i-1} + \epsilon_i, \quad i = 0, \pm 1, \pm 2, \ldots, \tag{2.1}$$

where α is a constant ($|\alpha| < 1$) and ϵ_i is a sequence of identically and independently distributed random variables. Gaver and Lewis (1980) found that for the sequence $\{X_i\}$ to have an exponential marginal distribution with parameter λ, the parameter α should be positive ($0 < \alpha < 1$) and that

$$X_i = \begin{cases} \alpha X_{i-1} \text{ w.p. } \alpha \\ \alpha X_{i-1} + E_i \text{ w.p. } 1 - \alpha \end{cases} \quad i = 0, \pm 1, \pm 2, \ldots, \tag{2.2}$$

where $\{E_i\}$ is an identically independent sequence of exponential marginal distribution with parameter λ.

The second order autoregressive model EAR(2) is defined as (Lawrance and Lewis (1980)):

$$X_i = \begin{matrix} \alpha_1 X_{i-1} \text{ w.p. } 1 - \alpha_2 \\ \alpha_2 X_{i-2} \text{ w.p. } \alpha_2 \end{matrix} + \epsilon_i, \tag{2.3}$$

where α_1 and α_2 are constants ($0 < \alpha_1, \alpha_2 < 1$). The basic feature of EAR(2) model is that X_i is always function of one of the previous two values X_{i-1} and X_{i-2}. In general the pth order model can be written as

$$X_i = \begin{matrix} \alpha_1 X_{i-1} & \text{w.p.} & a_1 \\ \alpha_2 X_{i-2} & \text{w.p.} & a_2 \\ \vdots & \vdots & \vdots \\ \alpha_p X_{i-p} & \text{w.p.} & a_p \end{matrix} + \epsilon_i \tag{2.4}$$

where

$$a_1 = (1 - \alpha_2), \qquad a_p = \prod_{j=2}^{p} \alpha_j$$

and

$$a_l = \prod_{j=2}^{l} \alpha_j(1 - \alpha_{l+1}), \quad l = 2, \ldots, p-1$$

3. ESTIMATION OF PARAMETERS IN EAR(2) PROCESS

The autocorrelation structure, the regression structure and the estimation of parameters in EAR(2) model have been discussed in this section.

Multiplying both sides of the equation (2.3) by X_{i-r} and taking expectations we have

$$E(X_i X_{i-r}) = (1 - \alpha_2)\{\alpha_1 E(X_{i-1}X_{i-r}) + E(X_{i-r})E(\epsilon_i)\} + \\ + \alpha_2\{\alpha_2 E(X_{i-2}X_{i-r}) + E(X_{i-r})E(\epsilon_i)\} \quad (3.1)$$

Substituting the expression for $E(\epsilon)$ given by

$$E(\epsilon) = (1 - \alpha_2)(1 - \alpha_1 + \alpha_2)E(X) \quad (3.2)$$

in Eqn. (3.1), the following is obtained

$$\varphi_r = \alpha_1(1 - \alpha_2)\varphi_{r-1} + \alpha_2^2\varphi_{r-2} \qquad (r = 1, 2, \ldots) \quad (3.3)$$

where

$$\varphi_r = \text{corr}(x_i, x_{i-r}) = \varphi_{-r} \quad \text{and} \quad \varphi_0 = 1$$

From the above equation the first two autocorrelations of the EAR(2) process can be obtained as follows:

$$\varphi_1 = \alpha_1/(1 + \alpha_2) \quad \text{and} \quad \varphi_2 = \alpha_1(1 - \alpha_2)\varphi_1 + \alpha_2^2 \quad (3.4)$$

Equation (3.4) can be solved to find the expressions for the parameters α_1 and α_2:

$$\alpha_1 = \{1 + (\varphi_2 - \varphi_1^2)/(1 - \varphi_1^2)\}^{1/2}\varphi_1 \quad \text{and} \\ \alpha_2 = \{(\varphi_2 - \varphi_1^2)/(1 - \varphi_1^2)\}^{1/2} \quad (3.5)$$

from (2.3)

$$E(X_i/X_{i-1} = x_{i-1}, X_{i-2} = x_{i-2}) = (1 + \alpha_2)\alpha_1 x_{i-1} + \alpha_2^2 x_{i-2} + \\ \tfrac{1}{\lambda}(1 - \alpha_2)(1 - \alpha_1 + \alpha_2) \quad (3.6)$$

which shows that the regression of X_i on X_{i-1} and X_{i-2} is linear in the given values x_{i-1} and x_{i-2} of X_{i-1} and X_{i-2}. The expected value is given by

$$E\{E(X_i/X_{i-1} = x_{i-1}, X_{i-2} = x_{i-2})\} = (1 - \alpha_2)\alpha_1 E(x_{i-1}) + \alpha_2^2 E(x_{i-2}) + \\ + \tfrac{1}{\lambda}(1 - \alpha_2)(1 - \alpha_1 + \alpha_2) = \tfrac{1}{\lambda}$$

since X_i follows exponential distribution with mean $\frac{1}{\lambda}$.

Estimation of parameters in second order autoregressive models has been attempted using (3.6) using the principle of least squares:

Let

$$S = \sum_{i=1}^{n} [y_i - \beta_1 x_{i-1} - \beta_2 x_{i-2} - \lambda^{-1}(1 - \beta_1 - \beta_2)]^2 \tag{3.7}$$

where $\beta_1 = (1 - \alpha_2)\alpha_1$ **and** $\beta_2 = \alpha_2^2$ **and** $y_i = E\{X_i | X_{i-1} = x_{i-1}\ X_{i-2} = x_{i-2}\}$.
Differentiating S **partially w.r.t.** β_1 **and aquating to zero we get**

$$\frac{\partial S}{\partial \beta_1} = 0 = -2 \sum_{i=1}^{n} \left[y_i - \beta_1 x_{i-1} - \beta_2 x_{i-2} - \frac{1}{\lambda}(1 - \beta_1 - \beta_2) \right] \left(-x_{i-1} + \frac{1}{\lambda} \right)$$

or

$$\frac{n\bar{y}}{\lambda} - \frac{\beta_1 n\bar{x}_1}{\lambda} - \frac{\beta_2 n\bar{x}_2}{\lambda} - S_{01} + \beta_1 S_{11} + \beta_2 S_{12} = 0$$

where

$$\bar{y} = \frac{\sum y_i}{n}, \qquad \bar{x}_1 = \frac{\sum x_{i-1}}{n}, \qquad \bar{x}_2 = \frac{\sum x_{i-2}}{n}$$

$$S_{01} = \sum y_i x_{i-1}, \qquad S_{11} = \sum x_{i-1}^2, \qquad S_{12} = \sum x_{i-1} x_{i-2}$$

or

$$\frac{n\bar{y}}{\lambda} - \frac{\beta_1 n\bar{x}_1}{\lambda} - \frac{\beta_2 n\bar{x}_2}{\lambda} - S_{01} + \beta_1 S_{11} + \beta_2 S_{12} = 0$$

We know that $\bar{y} = E(y_i) = \frac{1}{\lambda}$ and $\bar{x}_1 = E(x_{i-1}) = \bar{x}_2 = E(x_{i-2}) = \frac{1}{\lambda}$. Hence

$$\frac{n}{\lambda^2} - \frac{n\beta_1}{\lambda^2} - \frac{n\beta_2}{\lambda^2} - S_{01} + \beta_1 S_{11} + \beta_2 S_{12} = 0$$

or

$$\beta_1\left(S_{11} - \frac{n}{\lambda^2}\right) + \beta_2\left(S_{12} - \frac{n}{\lambda^2}\right) = \left(S_{01} - \frac{n}{\lambda^2}\right) \tag{3.8}$$

Similarly differentianting S with respect to β_2 and equating to zero, we have

$$\beta_1\left(S_{12} - \frac{n}{\lambda^2}\right) + \beta_2\left(S_{22} - \frac{n}{\lambda^2}\right) = \left(S_{02} - \frac{n}{\lambda^2}\right) \tag{3.9}$$

where

$$S_{22} = \sum_i x_{i-2}^2, \qquad S_{02} = \sum_i y_i x_{i-2}.$$

Solving equation (3.8) and (3.9) the following estimates for β_1 and β_2 are obtained

$$\hat{\beta}_1 = \frac{\left(S_{02} - \frac{n}{\lambda^2}\right)\left(S_{12} - \frac{n}{\lambda^2}\right) - \left(S_{01} - \frac{n}{\lambda^2}\right)\left(S_{22} - \frac{n}{\lambda^2}\right)}{\left(S_{12} - \frac{n}{\lambda^2}\right)^2 - \left(S_{11} - \frac{n}{\lambda^2}\right)\left(S_{22} - \frac{n}{\lambda^2}\right)} \tag{3.10}$$

$$\hat{\beta}_2 = \frac{\left(S_{01} - \frac{n}{\lambda^2}\right)\left(S_{12} - \frac{n}{\lambda^2}\right)\left(S_{02} - \frac{n}{\lambda^2}\right)\left(S_{11} - \frac{n}{\lambda^2}\right)}{\left(S_{12} - \frac{n}{\lambda^2}\right)^2 - \left(S_{11} - \frac{n}{\lambda^2}\right)\left(S_{22} - \frac{n}{\lambda^2}\right)} \tag{3.11}$$

Since $\hat{\beta}_1 = (1 - \alpha_2)\alpha_1$ and $\hat{\beta}_2 = \alpha_2^2$, the estimates for the parameters α_1 and α_2 in the EAR(2) process are given by

$$\hat{\alpha}_1 = \frac{\hat{\beta}_1}{(1 - \sqrt{\hat{\beta}_2})} \quad \text{and} \quad \hat{\alpha}_2 = \sqrt{\hat{\beta}_2} \tag{3.12}$$

4. CONCLUSIONS

The estimations of parameters in second order exponential autoregressive models has been attempted. There is scope for extending this idea for estimation of parameters in higher order exponential autoregressive models also.

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