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NOTAS

VARIANTS OF THE TIME MINIMIZATION ASSIGNMENT PROBLEM

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ABSTRACT

The present paper develops techniques to solve two variants of the time minimizing assignment problem. In the first, there are n jobs to be assigned to m establishmets ($m < n$) in such a way that the time taken to complete all the jobs is the minimum, it being assumed that all the jobs are commenced simultaneously. The second variant is an extension of the first one in the sense that an additional constraint on the minimum number of jobs to be taken up by each establishment is introduced. Numerical examples are included to illustrate the techniques.

1. INTRODUCTION

The classical assignment problem consists of assigning n jobs to an equal number of establishmets, one each, so as to meet certain objectives which may be the minimization of the cost incurred or the time taken to complete these jobs. There exist many practical situations where the ideal conditions do not exist and a certain number of jobs therefore may be required to be handled by a lesser number of establishments. For major projects, certain well equipped and resourceful establishmets may opt to undertake more than one job and in such situations, the decision maker may have to assign the jobs in the best possible manner to meet his objectives. This and many other similar situations give rise to the variant of the classical assignment problem being discussed in the paper. As already mentioned, the present paper concentrates on the time factor and the techniques developed, minimize the time of completion of all jobs, the basic assumption being that all jobs are commenced simultaneously.

The standard time minimizing assignment problem when $m = n$ is

$$\text{Minimize } [\text{Max}(t_{ij} | x_{ij} = 1)]$$

Subject to

$$\left.\begin{aligned} &\sum_{j=1}^{n} x_{ij} = 1, \quad i = 1 \text{ to } n \\ &\sum_{i=1}^{n} x_{ij} = 1, \quad j = 1 \text{ to } n. \\ &x_{ij} = 0,1. \end{aligned}\right\} (P)$$

Here t_{ij} is the time taken by the ith establishment to complete the jth job, x_{ij} is the decision variable associated with the pair (i,j) and in fact $x_{ij} = 1$ if the jth job is assigned to the ith establishment, $x_{ij} = 0$ otherwise.

Solution procedures for the above problem have been discussed in [1, 2].

Section I

Consider the situation where there are «m» establishments each competent enough to take up more than one job and the number of jobs to be completed in «n». The problem of the decision maker is to assign the jobs to the establishments in such a way that all the jobs are completed in minimum time; ti being assumed that each job is to be completed by one establishment only. The mathematical model of this problem is as follows:

$$\text{Minimize } [\text{Max}(t_{ij} | x_j = 1)]$$

Subject to

$$\begin{aligned} &\sum_{i \in I} x_{ij} = 1, \quad j \in J \\ &\sum_{j \in J} x_{ij} \geqslant 1, \quad i \in I \qquad\qquad (P_1) \\ &x_{ij} = 0,1 \end{aligned}$$

where $I = \{1, 2, \ldots, m\}$ is the index set of establishments and
$J = \{1, 2, \ldots, n\}$ is the index set of jobs.

The problem (P_1) is solved by constructing a classical assignment problem whose optimal solution provides an optimal solution of (P_1). Obviously, a necessary and sufficient condition for the problem (P_1) to have a solution is that $n \geqslant m$.

Consider now a standard time minimizing assignment problem defined as

$$\text{Minimize} \quad \text{Max}[\bar{t}_{ij} \mid y_{ij} = 1]$$

such that

$$\begin{aligned} &\sum_{i=1}^{n} y_{ij} = 1, \qquad j = 1 \text{ to } n \\ &\sum_{j=1}^{n} y_{ij} = 1, \qquad i = 1 \text{ to } n \qquad\qquad (P_2) \\ &y_{ij} = 0, 1. \end{aligned}$$

where

$$\begin{aligned} \bar{t}_{ij} &= t_{ij}, & (i,j) &\in I \times J \\ \bar{t}_{m+k,j} &= \min_{i \in I} t_{ij}, & k &= 1, 2, \ldots, n - m \\ & & j &\in J. \end{aligned}$$

The optimal solution of problem (P_1) can then be derived from an optimal solution of (P_2), as the following theorem asserts:

Theorem 1. The optimal objective values of problems (P_1) and (P_2) are equal.

PROOF. Let T_B be the optimal time for the problem (P_2) and T_A the same for problem (P_1).

Let if possible $T_A < T_B$.

For an optimal solution of (P_1), let J_i be the index set of the jobs undertaken by the ith establishment with $O(J_i) = m_i$.

Thus

$$\sum_{i \in I} m_i = n.$$

Let

$$J_k = \{j_k^{(1)}, j_k^{(2)}, \ldots, j_k^{(m_k)}\}.$$

An assignment for problem (P_2) corresponding to the above optimal solution of (P_1) is the given by

$$Y_{ij_i^{(1)}}(-1) = 1, \quad i \in I.$$

$$Y_m + \sum_{i=1}^{s-1} m_i + k, j_{k+s}^{(s)} = 1, \quad k = -(s-2), -(s-3), \ldots$$
$$, -(s-s), 1, 2, 3, \ldots, m_s - s.$$

Also, by virtue of the definition of $\bar{t}_{m+k,j}$, $k = 1$ to $n - m$, for this solution $\text{Max}[\bar{t}_{ij}\colon \bar{x}_{ij} = 1] \leqslant T_A < T_B$.

This contradicts the optimal character of T_B. Hence T_A cannot be less than T_B. Thus $T_A \geqslant T_B$.

Now suppose that $T_B < T_A$. Then starting from an optimal assignment of (P_2), an assignment of (P_1) can be derived as follows:

If

$$y_{m+k,j} = 1 \quad \text{and} \quad \bar{t}_{m+k,j} = t_{sj} = \min_{i \in I} t_{ij},$$

then set

$$x_{sj} = 1, \qquad k = 1 \quad \text{to} \quad n - m$$

and

$$x_{ij} = y_{ij} = 1 \qquad i \in I, j \in J.$$

The time associated with this solution is $T_B < T_A$. This contradicts the optimal character of T_A.

Thus

$$T_A = T_B.$$

Corolary. An optimal assignment of problem (P_2) provides an optimal assignment for (P_1).

The proof of this corollary follows directly form Theorem 1.

Procedure I. Formulate and solve problem (P_2). Let it's optimal assignment be $Y = (y_{ij})$.

Then the optimal assignment $X = (x_{ij})$ for (P_1) is given by

$$x_{ij} = y_{ij} = 1, \quad (i, j) \in I \times J.$$
$$x_{sj} = 1, \quad \text{if} \quad y_{m+k,j} = 1, \text{ and } t_{m+k,j} = t_{sj} = \min_{i \in I} t_{ij}, \quad k = 1 \quad \text{to} \quad n - m.$$

NUMERICAL

Consider a 4 × 7 asignment problem given by the following tableau.

2	1	3	2	4	2	2
4	3	1	1	2	3	4
1	3	4	2	2	3	1
1	2	1	4	3	2	1

The figure in each cell (i, j) gives t_{ij}.
Problem (P_2) then takes the following form:

2	1	3	2	4	2	2
4	3	1	1	2	3	4
1	3	4	2	2	3	1
1	2	1	4	3	2	1
1	1	1	1	2	2	1
1	1	1	1	2	2	1
1	1	1	1	2	2	1

An optimal solution of problem (P_2) is shown in the following tableau:

2	①	3	2	4	2	2
4	3	1	①	2	3	4
①	3	4	2	2	3	1
1	2	①	4	3	2	1
1	1	1	1	②	2	1
1	1	1	1	2	②	1
1	1	1	1	2	2	①

The solution is $y_{12} = 1$, $y_{24} = 1$, $y_{31} = 1$, $y_{43} = 1$, $y_{55} = 1$, $y_{66} = 1$, $y_{77} = 1$, all other y_{ij}'s are zero.

The time associated with this solution is $T = 2$.

Thus $T = 2$ is the optimal time of the original problem (P_1) and the corresponding solution as obtained by Procedure I is $x_{12} = x_{16} = 1$, $x_{24} = x_{25} = 1$, $x_{31} = x_{37} = 1$, $x_{43} = 1$, all other x_{ij}'s $= 0$.

Section 2

This section discusses an extension of the problem discussed in section I, where, in the present case, each establishment is requiered to do a specified minimum number of jobs. Thus if α_1 is the minimum number of jobs required to be done by the ith establisment, the mathematical formulation of the problem becomes

Minimize

$$[\mathrm{Max}(t_{ij} | x_{ij} = 1)]$$

Subject to

$$\begin{aligned} &\sum_{i \in I} x_{ij} = 1, && j \in J \\ &\sum_{j \in J} x_{Ij} \geqslant \alpha_i, && i \in I \\ &x_{ij} = 0,1 && (i,j) \in I \times J. \end{aligned} \qquad (P_3)$$

Here

$$\sum_{i \in I} \alpha_i \leqslant n.$$

Since the ith establishment is required to do at least α_i, jobs, $i \in I$, it may be thought of as consisting of α_i separate units of equal competence, each required to take up at least one job. The case $\sum_{i \in I} \alpha_i = n$ results in an $n \times n$ assignment problem (in terms of the units and jobs).

In case

$$\sum_{i=1}^{m} \alpha_i < n,$$

it reduces to the problem discussed in section I. Thus to solve problem (P_3), procedure I is modified as procedure II given below.

Procedure II. Extend the given assignment problem as follows: Write the row corresponding to the ith establishment α_i, times, $i = 1$ to m.

In case

$$\sum_{i=1}^{m} \alpha_i = n,$$

solve the $n \times n$ assignment problem. This provides a solution of (P_3) as given below:

If $\sum_{i \in I} \alpha_i = p < n$, add n-p fictitious rows with

$$t_{p+k,j} = \min_{i \in I} t_{ij}, \quad k = 1 \quad \text{to} \quad n - p.$$

Solve this $n \times n$ time assignemt problem. Then an optimal assignment for problem P_3 can be derived from this optimal solution as explained in Procedure I.

Thus often alloting exactly α_i jobs to the ith establishment the procedure allots $n - p$ jobs in the best possible way so that the time of completion of all the jobs is the least.

NUMERICAL

Consider the 2×6 assignment problem (P_3) given by the following table:

2	3	6	5	1	3
4	1	2	3	4	3

Let $\alpha_1 = 2$, $\alpha_2 = 3$ thus

$$\sum_{i=1}^{2} \alpha_i = 5 < 6 \text{ (the number of jobs).}$$

Extending the given assignment problem as given in procedure II, the following 5×6 extended problem (E) is obtained as follows:

2	3	6	5	1	3
2	3	6	5	1	3
4	1	2	3	4	3
4	1	2	3	4	3
4	1	2	3	4	3

Since

$$\sum_{i=1}^{2} \alpha_i = 5 < 6,$$

thus $6 - 5 = 1$ fictitious row has to be a added, according to Procedure II again. Thus a 6×6 assignment problem (EP) is obtained as follows:

2	3	6	5	1	3
2	3	6	5	1	3
4	1	2	3	4	3
4	1	2	3	4	3
4	1	2	3	4	3
2	1	2	3	1	3

An optimal solution to (EP) is given by

$y'_{11} = 1, y'_{22} = 1, y'_{33} = 1, y'_{44} = 1, y'_{56} = 1, y'_{65} = 1$ with associated time $T = 3$.

The corresponding optimal solution to problem (E) is therefore $y_{11} = y_{15} = 1$, $y_{22} = 1$, $y_{33} = 1$, $y_{44} = 1$, $y_{56} = 1$, with time $T = 3$. Thus an optimal solution to the original problem (P_3) is $x_{11} = x_{15} = x_{12} = 1$, $x_{23} = x_{24} = x_{26} = 1$, and the optimal time is $T = 3$.

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