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DISCRETE GENERALIZED LIOUVILLE-TYPE DISTRIBUTION AND RELATED MULTIVARIATE DISTRIBUTIONS

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SUMMARY

Discrete analogue of the Liouville distribution is defined and is termed as Discrete generalized Liouville-Type distribution (DGL-TD) Firstly, properties in its factorial and ordinary moments are given. Then by finding the covariance matrix, partial and multiple correlations for DGL-TD are evaluated. Multinomial, multivariate negative binomal and multivariate log series distributions are shown as particular cases of this general distribution. The asymptotic distribution of the estimates of the parameters is also attempted.

Key words

Liouville distribution; multivariate distributions; multinomial; log series; negative binomal.

AMS classification

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1. INTRODUCTION

Generalized Liouville distribution (GLD) of both I and II kind, as a multivariate distribution in $x_1, \dots, x_n$, with $x_i \geqslant 0$, $\sum_{i=1}^{n} x_i \leqslant 1$, and $x_1 + \dots + x_n$ has been discussed in Sivazlian (1981*a*, *b*). Some properties of GLD as related to Dirichlet distribution, have been given in Sivazlian (1981*b*). All of this discussion is restricted to continuous variables, $x_1, \dots, x_n$. What is being done here, is to give a discrete analogue of this distribution and this is termed as Discrete Generalized Liouville-Type distribution (DGL-TD). By evaluating the covariance matrix of this general distribution, partial and multiple correlations are evaluated and these are checked against the corresponding quantities in the multinomial, multivariate negative binomial (MNB) and in multivariate log series (MLS) distributions. Again, in order to determine the asymptotic distributions of the estimates of parameters, correlation and covariance matrix are evaluated.

2. DISCRETE GENERALIZED LIOUVILLE-TUPE DISTRIBUTION (DGL-TD)

Define

$$f(x) = \mathrm{A}g(s_n)(\phi\theta_1)^{x_1} \dots)\phi\theta_n)^{x_n}/x_1! \dots x_n! \tag{1}$$

$$\theta_i > 0,\ x_i \geqslant 0,\ x_i = 0, 1, 2, \dots,\ i = 1, 2, \dots, n,\ \theta = \theta_1 + \dots + \theta_n,$$

$$s_n = x_1 + \dots + x_n, \quad \phi = \phi(\theta). \quad 0 < \theta < 1$$

Setting

$$y_i = x_1 + \dots + x_i, \qquad i = 1, 2, \dots, n \tag{2}$$

it is easy to see from (1) that

$$A = 1 \Big/ \sum_{z=0}^{\infty} g(z)\theta^z/z! \tag{3}$$

and from (1), we have the mgf as

$$m(t) = A \sum g(z)[\phi(\theta_1 e^{t_1} + \dots + \theta_n e^{t_n})]^z/z! \tag{4}$$

$\sum$ denotes throughout this paper $\sum_0^\infty$ unless otherwise defined.

If

$$\prod_{i=1}^{n} x_i(x_i - 1)\ldots(x_i - s + 1) = [x_1, \ldots, x_n]^{(s)} \tag{5}$$

we have

$$E[x_1, \ldots, x_n]^{(s)} = \phi^{ns}(\theta_1, \ldots, \theta_n)^s A \Sigma g(z + ns)(\phi\theta)^z/z! \tag{5a}$$

and

$$E[x_i^{(s)}] = \mu'_{[s]i} = \phi^s \theta_i^s A \Sigma g(z + s)(\phi\theta)^z/z! \tag{6}$$

where $\mu'_{[s]i}$ denotes the s-th factorial moment of x_i. From (6), we get

$$(\phi\theta_i)^{-s}\mu'_{[s]i} = 1 + \sum_{i=0}^{s-1} A\ \Sigma[\Delta g(z + i)](\phi\theta)^z/z! \tag{7}$$

where $\Delta g(z + i) = g(z + i + 1) - g(z + i)$, and also

$$A \sum_{z=0}^{\infty} [\Delta^r g(z)](\phi\theta)^z/z! = \sum_{s=0}^{r} \binom{r}{s} \mu'_{[s]i}(-1)^s(\phi\theta_i)^{-s} \tag{8}$$

with $\mu'_{[0]i} = 1$.

From (4), we get with

$$\begin{aligned} B &= \phi A \Sigma g(z + 1)\theta^z/z! \\ C &= \phi^2 A \Sigma g(z + 2)\theta^z/z! \end{aligned} \tag{9}$$

that

$$\begin{aligned} E(x_i) &= B\theta_i \\ E[x_i(x_i - 1)] &= c\theta_i^2 \\ E[x_i x_j] &= c\theta_i\theta_j \end{aligned} \tag{10}$$

and from (10), we have

$$\begin{aligned} \sigma_i^2 &= \theta_i(B + C\theta_i) - B^2\theta_i^2 \\ \sigma_{ij} &= C\theta_i\theta_j - B^2\theta_i\theta_j \end{aligned} \tag{11}$$

From (11), we can write the covariance matrix Ω as

$$\Omega = E + ed' \tag{12}$$

where E is the diagonal matrix with the diagonal elements $e_{ii} = (B + C\theta_i)\theta_i$ and non-diagonal elements $e_{ij} = C\theta_i\theta_j$, $i, j = 1, 2, \ldots, n$ and vectors $e' = (-B\theta_1, \ldots, -B\theta_n)$ and $d' = (B\theta_1, \ldots, B\theta_n)$.

Now

$$|\Omega| = |E|[1 + d'E^{-1}e] \tag{13}$$

and

$$|E| = (\theta_1, \ldots, \theta_n)B^{n-1}(B + C\theta) \tag{13a}$$

In the matrix E^{-1}, diagonal elements $\bar{e}_{ii} = [B + C(\theta - \theta_i)]B\theta_i(B + C\theta)$ and *all* non-diagonal elements $\bar{e}_{ij} = -C/B(B + C\theta)$, $i, j = 1, 2, \ldots, n$ and hence

$$d'E^{-1}e = -B^2\theta/(B + C\theta) \tag{13b}$$

and hence

$$|\Omega| = (\theta_1, \ldots, \theta_n)B^{n-1}[B + \theta(C - B^2)] \tag{14}$$

and similarly

$$\Omega_{11} = E_{11} + e_1 d'_1 \tag{15}$$

where E_{11} is the matrix E with first row and first column suppressed and vectors e'_1, d'_1 are respectively e', d' with their first elements missing. We have

$$|E_{11}| = [B + C(\theta - \theta_1)]B^{n-2}(\theta_2, \ldots, \theta_n) \tag{15a}$$

In the inverse matrix E_{11}^{-1}, diagonal elements $b_{ii} = [B + C(\theta - \theta_1 - \theta_i)]/B\theta_i[B + C(\theta - \theta_1)]$ and *all* the non-diagonal elements $b_{ij} = C/B[B + C(\theta - \theta_1)]$, $i, j = 2, 3, \ldots, n$ and hence

$$d'_1 E_{11}^{-1} e_1 = -B^2(\theta - \theta_1)/[B + C(\theta - \theta_1)] \tag{15b}$$

and we have

$$|\Omega_{11}| = (\theta_2, \ldots, \theta_n)B^{n-2}[B + (\theta - \theta_1)(C - B^2)] \tag{16}$$

and from (11) we have

$$\sigma_1^2 = \theta_1[B + \theta_1(C - B^2)]. \tag{17}$$

Now from (14), (16) and (17), we have the multiple correlation

$$1 - R^2_{1(2,3,\ldots,n)} = \frac{B[B + \theta(C - B^2)]}{[B + (\theta - \theta_1)(C - B^2)]\,[B + \theta_1(C - B^2)]} \tag{18}$$

Now similarly, we have

$$\Omega_{12} = E_{12} + e_2 d_2' \tag{19}$$

where E_{12} is the matrix E with its first row and second column missing and the vectors

$$e_2' = (-B\theta_2, -B\theta_3, \ldots, -B\theta_n), \qquad d_2' = (B\theta_1, B\theta_3, \ldots, B\theta_n) \tag{19a}$$

we have

$$|E_{12}| = (\theta_1, \ldots, \theta_n)CB^{n-2} \tag{19b}$$

and

$$|\Omega_{12}| = |E_{12}|[1 + d_2' E_{12}^{-1} e_2] \tag{19c}$$

let

$$E_{12}^{-1} = \begin{bmatrix} C_{21} & C_{23} & \ldots & C_{2n} \\ C_{31} & C_{33} & \ldots & C_{3n} \\ \vdots & & & \vdots \\ C_{n1} & C_{n3} & \ldots & C_{nn} \end{bmatrix} \tag{19d}$$

Using (19*b*), we have

$$\begin{aligned} C_{21} &= [B + C(\theta - \theta_1 - \theta_2)]/BC\theta_1\theta_2 \\ C_{ii} &= 1/B\theta_i, \quad i = 3, 4, \ldots, n \\ C_{2i} &= -1/B\theta_1, \quad i = 3, 4, \ldots, n \\ C_{i1} &= -1/B_2, \quad i = 3, 4, \ldots, n \end{aligned} \tag{19e}$$

and all other elements vanish. That is, in E_{12}^{-1}, we have first row, first column and the diagonal elements only. All other elements are zero. Then from (19*c*) and (19*e*), we have

$$|\Omega_{12}| = (\theta_1, \ldots, \theta_n)B^{n-2}(C - B^2). \tag{19f}$$

Now from, $|\Omega_{12}|$, $|\Omega_{11}|$ and $|\Omega_{22}|$, we have the partial correlations as

$$\rho^2_{12.34,,\ldots,n} = \frac{\theta_1\theta_2[C - B^2]^2}{[B + (\theta - \theta_1)(C - B^2)]\,[B + (\theta - \theta_2)(C - B^2)]} \tag{20}$$

$\rho^2_{12.34,\ldots,n} \leqslant 1$ implies

$$(C - B^2)(\theta_1 + \theta_2) \leqslant B + \theta(C - B^2). \tag{20a}$$

Equation (20*a*) implies

$$\begin{array}{ll} \theta - \theta_1 - \theta_2 \leqslant 1 & \text{for multinomial.} \\ \theta_1 + \theta_2 \leqslant 1 & \text{for MNB and} \\ (\theta_1 + \theta_2)(a - 1) \leqslant (a - \theta) & \text{for MLS.} \end{array}$$

From (18) it follows

$$u = B + \theta(C - B^2) \tag{20b}$$

and

$$v = B + (\theta - \theta_1)(C - B^2) \tag{20c}$$

have the same sign. From $\rho^2_{12.34,\ldots,n} \geqslant 0$ along with (20*c*) implies

$$w = B + (\theta - \theta_2)(C - B^2) \tag{20d}$$

has the same sign as u and v. In the multinomial case since $u = k(1 - \theta)$ which is > 0, it follows for this case that both $v > 0$ and $w > 0$ which means $1 + \theta_1 > \theta$ and $1 + \theta_2 > \theta$ which of course is true. For the case of MNB $u > 0$, $v > 0$, $w > 0$ imply respectively $0 < 1$, $\theta_1 < 1$ and $\theta_2 < 1$. For MLS $n > 0$ implies $a > \theta$ which means $1 < \theta + e^{-\theta}$. Similarly $v > 0$ for MLS implies $a > (\theta - \theta_1)/(1 - \theta_1)$ which in turn along with $a > \theta$ says $1 < \theta + \exp[\theta_1(1 + \theta)/2(1 - \theta_1)]$ which is true.

$1 - R^2_{1(23,\ldots,n)} \leqslant 1$ implies

$$\theta_1(\theta - \theta_1)[C - B^2]^2 \geqslant 0 \tag{20e}$$

which is true. Note $(C - B^2)$ may be negative as in the case of multinomial.

3. SPECIAL CASES

3a. Multinomial

$$f(x) = \theta_1^{x_1}, \ldots, \theta_n^{x_n}(1-\theta)^{k-s_n}k!/(x_1!, \ldots, x_n!)(k-s_n)! \tag{21}$$

$$\theta_i > 0, \quad \Sigma\theta_i < 1, \quad x_i = 0, 1, 2, \ldots, \quad i = 1, 2, \ldots, n$$

$$\sum_{i=1}^{n} x_i = s_n \leqslant k, \quad \theta = \theta_1 + , \ldots, + \theta_n.$$

We have for this case

$$\begin{aligned} g(z) &= 1/(k-z)!, \quad \phi(\theta) = 1/(1-\theta) \\ A &= (1-\theta)^k k!, \quad B = k, \quad C = k(k-1) \end{aligned} \tag{21a}$$

and hence

$$\begin{aligned} &E(\theta_i) = k\theta_i \\ &\sigma_i^2 = k\theta_i(1-\theta_i) \\ &\sigma_{ij} = -k\theta_i\theta_j \end{aligned} \tag{21b}$$

$$|\Omega| = (\theta_1, \ldots, \theta_n)k^n(1-\theta)$$

$$|\Omega_{11}| = (\theta_2, \ldots, \theta_n)k^n(1-\theta+\theta_1) \tag{22a}$$

$$1 - R^2_{1(23,\ldots,n)} = \frac{1-\theta}{(1-\theta+\theta_1)(1-\theta_1)} \tag{23}$$

$$\rho^2_{12.34,\ldots,n} = \frac{\theta_1\theta_2}{(1-\theta+\theta_1)(1-\theta+\theta_2)} \tag{23a}$$

$1 - R^2 \leqslant 1$ implies $\theta \geqslant \theta_1$ which is true and $\rho^2_{12.34,\ldots,n} \leqslant 1$ implies

$$1 + \theta_1 + \theta_2 > 0 \tag{23b}$$

3b. Multivariate negative binomial (MNB)

$$f(x) = \frac{\Gamma(k+s_n)\theta_1^{x_1}, \ldots, \theta_n^{x_n}(1-\theta)^k}{\Gamma(k)x_1!, \ldots, x_n!} \tag{24}$$

$$0 < k < \infty, \quad 0 < \theta_i < 1, \quad \theta_1 + \ldots + \theta_n < 1,$$

$x_i = 0, 1, 2, \ldots \quad i = 1, 2, \ldots, n.$

Now for MNB, we have

$$\begin{aligned} &g(z) = \Gamma(k+z), \quad \phi = 1 \\ &A\Gamma(k)/(1-\theta)^k, \quad B = k/(1-\theta), \quad C = k(k+1)/(1-\theta)^2 \end{aligned} \tag{24b}$$

and

$$\begin{aligned}
\mu_i &= E(x_i) = k\theta_i/(1-\theta) \\
\sigma_i^2 &= k\theta_i(1-\theta+\theta_i)/(1-\theta)^2 \qquad (24c) \\
\sigma_{ij} &= k\theta_i\theta_j/(1-\theta)^2 \\
|\Omega| &= (\theta_1 \ldots, \theta_n)k^n/(1-\theta)^{n+1} \\
|\Omega_{11}| &= (\theta_2, \ldots, \theta_n)k^n(1-\theta_1)/(1-\theta)^n \qquad (24d) \\
\sigma_1^2 &= r\theta_1(1-\theta+\theta_1)/(1-\theta)^2
\end{aligned}$$

and hence

$$1 - R^2_{1(23,\ldots,n)} = (1-\theta)/(1-\theta+\theta_1)(1-\theta_1) \qquad (25)$$

and

$$\rho^2_{12.34,\ldots,n} = \theta_1\theta_2/(1-\theta_1)(1-\theta_2) \qquad (25a)$$

equations (24*d*), (25) and (25*a*) chech with those corresponding equations in Lingappaiah (1982).

3c. Multivariate log series (MLS)

$$f(x) = \Gamma(s_n)\theta_1^{x_1}, \ldots, \theta_n^{x_n}/(x_1!, \ldots, x_n!)[-\log(1-\theta)] \qquad (26)$$
$$x_i = 0, 1, 2, \ldots, \quad 0 < \theta_i < 1, \quad i = 1, 2, \ldots, n, \quad s_n > 0, \quad \theta < 1$$

Here we have

$$\phi = 1, \quad g(z) = z! \quad A = \tfrac{1}{a} = [-\log(1-\theta)]^{-1} \qquad (26a)$$
$$B = 1/(1-\theta)a, \quad C = 1/a(1-\theta)^2, \quad \mu_i = \theta_i/a(1-\theta)$$
$$\sigma_i^2 = C\theta_i^2 + B\theta_i - B^2\theta_i^2, \quad \sigma_{ij} = C\theta_i\theta_j - B^2\theta_i\theta_j \qquad (26b)$$

and

$$|\Omega| = (\theta_1, \ldots, \theta_n)(a-\theta)/[a(1-\theta)]^{n+1} \qquad (26c)$$
$$|\Omega_{11}| = (\theta_2, \ldots, \theta_n)[a(1-\theta_1) - (\theta-\theta_1)]/[a(1-\theta)]^n \qquad (26d)$$
$$\sigma_1^2 = \theta_1[(a-\theta_1) - a(\theta-\theta_1)]/a^2(1-\theta)^2 \qquad (26e)$$
$$1 - R^2_{1(23,\ldots,n)} = \frac{a(1-\theta)(a-\theta)}{[a(1-\theta_1)-(\theta-\theta_1)][(a-\theta_1)-a(\theta-\theta_1)]} \qquad (27)$$

Items (26*b*) to (27) correspond to those items in Patil an Bildikar (1967). Also, we have

$$\rho^2_{12.34,\ldots,n} = \frac{\theta_1\theta_2(1-a)^2}{[a(1-\theta_1)-(\theta-\theta_1)][a(1-\theta_2)-(\theta-\theta_2)]} \qquad (28)$$

4. ASYMPTOTIC DISTRIBUTIONS

Below, we try to find the asymptotic distributions of θ_i's. For this purpose, expressions for $E(\partial^2 \log L/\partial\theta_i^2)$ and $E(\partial^2 \log L/\partial\theta_i\partial\theta_j)$ are obtained in general. Then the elements of covariance matrix are given in table I from which the asymptotic distributions can be obtained.

From (1), we have

$$\log f(x) = -\log[\Sigma g(z)\theta^z/z!] + \sum_{i=1}^{n} x_i \log\theta_i + s_n \log\phi + \psi(x) \tag{29}$$

where $\psi(x)$ is a function of $x_1, \ldots, x_n$ only. Now,

$$D_i = \frac{\partial}{\partial\theta_i}\log f = \frac{x_i}{\theta_i} + (s_n)(\phi'/\phi) - A\Sigma\frac{z(\theta\phi)^{z-1}g(z)(\phi + \theta\phi')}{z!} \tag{29a}$$

$$= \frac{x_i}{\theta_i} + (s_n)(\phi'/\phi) - B\left(1 + \frac{\theta\phi'}{\phi}\right) \tag{29b}$$

$$\begin{aligned} D_i^2 = \frac{\partial^2}{\partial\theta_i^2}\log f = & -\left(\frac{x_i}{\theta_i^2}\right) + (s_n)\left[\frac{\phi\phi'' - \phi'^2}{\phi^2}\right] \\ & + A^2[\Sigma z(\theta\phi)^{z-1}g(z)(\phi + \theta\phi')/z!]^2 \\ & - A[\Sigma z(z-1)(\theta\phi)^{z-2}g(z)(\phi + \theta\phi')^2/z! \\ & + \Sigma z(\theta\phi)^{z-1}g(z)(2\phi' + \theta\phi'')/z!] \end{aligned} \tag{29c}$$

$$= \frac{-x_i}{\theta_i^2} + (s_n)\left(\frac{\phi\phi'' - \phi'^2}{\phi^2}\right] - (C - B^2)\left(1 + \frac{\theta\phi'}{\phi}\right)^2 - \tag{29d}$$

$$- B\left(\frac{2\phi' + \theta\phi''}{\phi}\right) - E(D_i^2) = \left(\frac{\mu_i}{\theta_i^2}\right) + H(\theta) \tag{29e}$$

where

$$H(\theta) = B\left(\frac{2\phi' + \theta\phi''}{\phi}\right) - B\theta\left(\frac{\phi\phi'' - \phi'^2}{\phi^2}\right) + (C - B^2)\left(1 + \frac{\theta\phi'}{\phi}\right)^2 \tag{29f}$$

Similarly, we have

$$-E(D_{ij}^2) = -E\left(\frac{\partial^2}{\partial\theta_i\partial\theta_j}\log f\right) = H(\theta) \tag{29g}$$

Hence, the asymptotic distribution of $(\theta_1, \ldots, \theta_n)$ is

$$f(\theta_1, \ldots, \theta_n) = C_0 \exp\left\{ -\frac{n}{2}\left[\sum_i \sum_j \left(\frac{\delta_{ij}\mu_i}{\theta_i^2} + H(\theta) \right) (\hat{\theta}_i - \theta_i)(\hat{\theta}_j - \theta_j) \right] \right\}$$

where $\delta_{ij} = 1$ if $i = j$ and $\delta_{ij} = 0$ if $i \neq j$ and $C_0 = \{ |R|(n/2\pi)^n \}^{1/2}$ and $|R|$ is given in Table 1.

Comments

As can be seen in DGL-TD, there are two places for one to choose the functions. That is, $g(z)$ and $\phi(\theta)$. By the choice of these two, one can generate a large number of distributions. This choice also includes multivariate generalized negative binomial of Patel (1979) though $g(z)$ in this case is slightly complex because of b_i's. Also it includes some of the distributions given in Sibuya *et al.* (1964) and Janardan (1975). Our purpose here is to give some general expressions for such quantities as Ω, Ω_{11}, Ω_{12}, $R_{1(23,\ldots,n)}$ and $\rho_{12.34,\ldots,n}$ so that one could easily evaluate corresponding quantities for the distribution (1) with chosen $\phi(\theta)$ and $g(z)$. Again the comments made above with respect to $R^2_{1(23,\ldots,n)}$ and $\rho^2_{12.34,\ldots,n}$ also apply for section 4 where also general expressions for $|R|$ and V help to obtain asymptotic distribution of the estimates for corresponding choices of $g(z)$ and $\phi(\theta)$.

Table I

	$\lvert R \rvert$	i-th diagonal element in $V = R^{-1}$	ij-th non-diagonal element in $V = R^{-1}$
General (DGL-TD)	$\left[\prod_{i=1}^{n}\left(\frac{\mu_i}{\theta_i^2}\right)\right] \cdot \left[1 + H(\theta)\sum_{i=1}^{n}\left(\frac{\theta_i^2}{\mu_i}\right)\right]$	$\dfrac{1 + H(\theta)\sum_{\substack{j=1\\ j\neq i}}^{n}\left(\frac{\theta_j^2}{\mu_j}\right)}{\left(\frac{\mu_i}{\theta_i^2}\right)\left[1 + H(\theta)\sum_{i=1}^{n}\left(\frac{\theta_i^2}{\mu_i}\right)\right]}$	$\dfrac{-H(\theta)\theta_i^2\theta_j^2/\mu_j\mu_i}{1 + H(\theta)\sum_{i=1}^{n}\left(\frac{\theta_i^2}{\mu_i}\right)}$
Multinomial	$\dfrac{k^n}{(1-\theta)(\theta_1\theta_2, \ldots, \theta_n)}$ $\phi = 1/(1-\theta)$	$\dfrac{\theta_i(1-\theta_i)}{k}$	$\dfrac{-\theta_i\theta_j}{k}$
Multivariate Negative binomial (MNB)	$\dfrac{k^n}{(\theta_1, \ldots, \theta_n)(1-\theta)^{n+1}}$ $\phi = 1, \quad H(\theta) = C - B^2$	$\dfrac{\theta_i(1-\theta_i)(1-\theta)}{k}$	$\dfrac{-\theta_i\theta_j(1-\theta)}{k}$
Multivariate log series (MLS)	$\dfrac{a-\theta}{[a(1-\theta)^{n+1}(\theta_1, \ldots, \theta_n)}$ $\phi = 1, \quad \mathrm{H}(\theta) = C - B^2$ $a = -\log(1-\theta)$	$\dfrac{a\theta_i(1-\theta)[a(1-\theta) + (a-1)(\theta-\theta_i)]}{(a-\theta)}$	$\dfrac{-(a-1)\theta_i\theta_j a(1-\theta)}{a-\theta}$

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