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DISTRIBUTION OF RANGE AND QUASI-RANGE FROM DOUBLE TRUNCATED EXPONENTIAL DISTRIBUTION

P. C. Joshi
N. Balakrishnan*
Department of Mathematics
Indian Institute of Technology,
Kanpur-208016 (India)

Abstract

For a doubly truncated exponential distribution, the probability density function of a quasi-range is derived. From this the density of sample range is obtained as a special case. Expressions for the mean and variance of the range are also obtained.

1. INTRODUCTION

Let $X_1, X_2, \ldots, X_n$ be a random sample of size $n(\geqslant 2)$ drawn from a doubly truncated exponential distribution with cumulative distribution function (c.d.f.) $F(x)$ and probability density function (p.d.f.) $f(x)$ given by

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* N. Balakrishnan is currently at the Department of Mathematical Sciences, McMaster University, Hamilton, Ontario, Canada L8S 4K1.

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$$f(x) = \exp(-x)/(P - Q), \quad -\log(1 - Q) < x < -\log(1 - P), \qquad (1)$$

where Q and $1 - P$ $(Q < P)$ are respectively the proportions of truncation on left and right of the standard exponential distribution. Let $X_{1:n} \leqslant X_{2:n} \leqslant \ldots \leqslant X_{n:n}$ be the corresponding order statistics. Following JOSHI [1979], denote $-\log(1 - Q)$ by Q_1, $-\log(1 - P)$ by P_1, $(1 - Q)/(P - Q)$ by Q_2, and $(1 - P)/(P - Q)$ by P_2. For the left truncation case $P = 1$, the quantities P_1, P_2, and Q_2 reduce to ∞, 0, and 1 respectively.

Order statistics from this and right truncated exponential distribution have been considered by several authors in recent years, for example, see SALEH/SCOTT/JUNKINS [1975] and JOSHI [1978, 1979]. These authors have mentioned their usefulness in life testing experiments and have given expressions and recurrence relations for moments of order statistics. The range and quasi-ranges play an important role in several problems of statistical inference. For the untruncated case, the p.d.f. and moments of quasi-ranges are given by RIDER [1959]. Recently, MALIK [1980] has obtained their c.d.f. for logistic distribution, where several related references are also given. In this paper, we obtain the density of $W_{r,s:n} = X_{s:n} - X_{r:n} (1 \leqslant r < s \leqslant n)$ for random samples drawn from (1) and consider the range as a special case.

2. THE DENSITY OF A QUASI-RANGE

For the density of a quasi-range $W_{r,s:n}$, consider the joint p.d.f. of $X_{r:n}$ and $X_{s:n}$ given by [David, 1970, p. 9]

$$f_{r,s:n}(x, y) = C_{r,s,n}[F(x)]^{r-1}[F(y) - F(x)]^{s-r-1}[1 - F(y)]^{n-s}f(x)f(y),$$
$$Q_1 \leqslant x < y < P_1,$$

where

$$C_{r,s,n} = n!/[(r - 1)!\,(s - r - 1)!\,(n - s)!].$$

Making a transformation $w = y - x$, $x = x$, integrating out x, and substituting for $f(x)$, $F(x)$ etc. from equation (1), we get the marginal p.d.f. of $W_{r,s:n}$ as

$$h(w) = C_{r,s,n}(P - Q)^{-n} e^{-w}(1 - e^{-w})^{s-r-1} \times$$

$$\int_{Q_1}^{P_1 - w} [1 - Q - e^{-x}]^{r-1}[P - 1 + e^{-(x+w)}]^{n-s} e^{-x(s-r+1)} dx,$$

$$0 \leqslant w < P_1 - Q_1.$$

Writing w_1 for e^{-w}, and expanding $[P - 1 + e^{-(x+w)}]^{n-s}$ in powers of $(P - 1)$ and $\exp(-x - w)$, we have

$$h(w) = C_{r,s,n}(P - Q)^{-n}(1 - w_1)^{s-r-1} \sum_{i=0}^{n-s} \binom{n-s}{i} (P - 1)^i \times$$

$$w_1^{n-s-i+1} \int_{Q_1}^{P_1 - w} [1 - Q - e^{-x}]^{r-1} e^{-x(n-i-r+1)} dx.$$

Putting $e^{-x} = t(1 - Q)$ in the above integral, we get

$$h(w) = C_{r,s,n}(1 - w_1)^{s-r-1} \sum_{i=0}^{n-s} (-1)^i \binom{n-s}{i} P_2^i Q_2^{n-i} \times$$

$$w_1^{n-s-i+1} \int_{P_2/(w_1 Q_2)}^{1} t^{n-i-r}(1 - t)^{r-1} dt,$$

$$0 \leqslant w < P_1 - Q_1, \tag{2}$$

where $w_1 = e^{-w}$. Equation (2) gives the p.d.f. of $W_{r,s:n}$ in terms of incomplete beta integrals. Alternatively, since r is likely to be small, it often will pay to expand $(1 - t)^{r-1}$ in powers of t and integrate term by term. This yields the p.d.f. of $W_{r,s:n}$ as

$$h(w) = C_{r,s,n}(1 - w_1)^{s-r-1} \sum_{i=0}^{n-s} (-1)^i \binom{n-s}{i} P_2^i Q_2^{n-i} \times$$

$$w_1^{n-s-i+1} \sum_{j=0}^{r-1} (-1)^{r-1-j} \binom{r-1}{j} \times$$

$$[1 - \{P_2/(w_1 Q_2)\}^{n-i-j}]/(n - i - j). \tag{3}$$

For special cases, the density expression given at (3) can be simplified further. Thus, for the left truncation case $P = 1$, equation (2) gives

$$\begin{aligned} h(w) &= C_{r,s,n}(1 - w_1)^{s-r-1} w_1^{n-s+1} \int_0^1 t^{n-r}(1 - t)^{r-1} dt \\ &= (n - r)!/[(s - r - 1)!\,(n - s)!] \exp[-w(n - s + 1)] \times \\ &\quad [1 - \exp(-w)]^{s-r-1}, \quad 0 \leqslant w < \infty, \end{aligned}$$

which does not depend on the left truncation proportion Q and hence is identical with the density of $W_{r,s:n}$ in samples drawn from standard exponential distribution, for example, see RIDER [1959] and DAVID [1970, p. 20].

3. MEAN AND VARIANCE OF RANGE

For the remainder of this paper we confine our attention to the sample range $W \equiv W_{1,n:n}$ for the case $P < 1$. Setting $r = 1$ and $s = n$ in equation (3) we obtain the p.d.f. of W as

$$h(w) = (n-1)e^{-w}(1-e^{-w})^{n-2}[Q_2^n - P_2^n e^{nw}], \quad 0 \leqslant w < a,$$

where $a = P_1 - Q_1$ is the upper limit for W. Using a result given in PARZEN [1960, p. 212], we can write

$$E(W^k) = k\int_0^a w^{k-1}[1 - H(w)]\,dw, \tag{4}$$

where $H(w)$ is the c.d.f. of W given by

$$H(w) = Q_2^n(1-e^{-w})^{n-1} - P_2^n(e^w - 1)^{n-1}, \quad 0 \leqslant w < a. \tag{5}$$

Using equations (4) and (5), $E(W)$ can be obtained by direct integration and is given by

$$E(W) = a[1 - Q_2^n - (1-Q_2)^n] - \sum_{i=1}^{n-1}[Q_2^{n-i} - (1-Q_2)^{n-i}]/i. \tag{6}$$

As a check, note that from the recurrence relations for $E(X_{r:n})$ given by JOSHI [1979], it is easy to show that

$$E(X_{n:n}) = P_1 + \sum_{i=1}^{n} Q_2^{n-i}/i - Q_2^n(P_1 - Q_1),$$

$$E(X_{1:n}) = Q_1 + \sum_{i=1}^{n} (1-Q_2)^{n-i}/i + (1-Q_2)^n(P_1 - Q_1).$$

Now $E(W) = E(X_{n:n}) - E(X_{1:n})$ immediately establishes (6). Next, for $k = 2$, equations (4) and (5) give

$$E(W^2) = a^2 - 2Q_2^n I_1 + 2P_2^n I_2, \tag{7}$$

where

$$I_1 = \int_0^a w(1 - e^{-w})^{n-1}\,dw, \tag{8}$$

$$I_2 = \int_0^a w(e^w - 1)^{n-1}\,dw. \tag{9}$$

There are several ways of evaluating the integrals given at (8) and (9). The easiest is to expand the integrand binomially and integrate by parts. However, this introduces combinatorial terms which increase rapidly as n increases. Alternatively, we can write I_1 as

$$\begin{aligned} I_1 &= \int_0^a w[1 - \{1 - (1 - e^{-w})^{n-1}\}]\,dw \\ &= a^2/2 - \sum_{i=0}^{n-2} \int_0^a w(1 - e^{-w})^i e^{-w}\,dw. \end{aligned} \tag{10}$$

Integrating the integral appearing at (10) by parts we get

$$I_1 = a^2/2 - \sum_{i=0}^{n-2} (i+1)^{-1}[a(1 - e^{-a})^{i+1} - \int_0^a (1 - e^{-w})^{i+1}\,dw].$$

Using the same technique again and simplifying we get

$$\begin{aligned} I_1 = a^2/2 &+ a \sum_{i=0}^{n-2} (i+1)^{-1}[1 - Q_2^{-(i+1)}] \\ &- \sum_{i=0}^{n-2} (i+1)^{-1} \sum_{j=0}^{i} Q_2^{-(j+1)}/(j+1). \end{aligned}$$

Similarly I_2 is given by

$$\begin{aligned} (-1)^{n-1} I_2 = a^2/2 &- a \sum_{i=0}^{n-2} (i+1)^{-1}[1 - (1 - Q_2)^{-(i+1)}] \\ &- \sum_{i=0}^{n-2} (i+1)^{-1} \sum_{j=0}^{i} (1 - Q_2)^{-(j+1)}/(j+1). \end{aligned}$$

Substituting for I_1 and I_2 in (7) and noting that $a = P_1 - Q_1$, we finally get a finite series expression for $E(W^2)$ from which the variance of W can be obtained.

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