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A NOTE ON PÓLYA'S THEOREM

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Abstract

The class of extended PÓLYA functions $\Omega = \{\phi: \phi$ is a continuous real valued real function, $\phi(-t) = \phi(t) \leqq \phi(0) \in [0, 1]$, $\lim_{t\to\infty} \phi(t) = c \in$ $\in [0, 1]$ and $\phi(|t|)$ is convex$\}$ is a convex set. Its extreme points are identified, and using Choquet's theorem it is shown that $\phi \in \Omega$ has an integral representation of the form $\phi(|t|) = \int_0^\infty \max\{0, 1 - |t|y\}\, dG(y)$, where G is the distribution function of some random variable Y. As on the other hand $\max\{0, 1 - |t|\}$ is the characteristic function of an absolutely continuous random variable X with probability density function $f(x) = (2\pi)^{-1}(x/2)^{-2}\sin^2(x/2)$, we conclude that ϕ is the characteristic function of the absolutely continuous random variable $Z = XY$, X and Y independent. Hence any $\phi \in \Omega$ is a characteristic function. This proof sheds an interesting light upon PÓLYA's sufficient condition for a given function to be a characteristic function.

Key words: Convex characteristic function
Pólya's class
Choquet's representation
extreme points

(*) Recibido, Enero, 1982

1. Though there exist some necessary and sufficient conditions for a given function to be the characteristic function of some real random variable (cf. LUKACS, 1970), eventually with atoms of probability at $-\infty$ and/or at $+\infty$, these results seem to be, in general, completely inappropriate to decide whether a given function is, or is not, a characteristic function.

On the other hand, the sufficient condition for a given real valued real function to be a characteristic function due to PÓLYA (1949) —if ϕ is a continuous real valued real function such that $\phi(-t) = \phi(t) \leqq 1$, $\lim_{t\to\infty} \phi(t) = c \in [0, 1]$ and $\phi(|t|)$ is convex, then ϕ is a characteristic function— is easy to apply. Observe that in the original paper of PÓLYA (1949) there was the assumption $\lim_{t\to\infty} f(t) = 0$, but this isn't in fact necessary: in the above statement, if $\lim_{t\to\infty} \phi(t) = c \in [0, 1]$, there exists an atom of probability c at the origin (and if $\phi(0) < 1$, there exist atoms of probability $(1 - \phi(0))/2$ at $-\infty$ and at $+\infty$). Observe futher that the practical interest of PÓLYA's characteristic functions is rather limited, since they correspond to random variables without finite variance (an example of theoretically interesting PÓLYA's characteristic functions: symmetric stable characteristic functions with characteristic exponent α less than 1).

In the present paper we put forward a new proof of PÓLYA's theorem, by showing that any PÓLYA function, i.e. any function satisfying the assumptions in PÓLYA theorem, admits an integral representation of Choquet's type. In order to do so, we begin with some preliminaries on convexity, we identify the extreme points of the (convex) set of PÓLYA functions and, at the end, we exhibid the integral representation refered to above. The arithmetic properties of PÓLYA's class of characteristic functions appear in PESTANA (1979).

2. We shall say that the real valued real function f is convex on an interval I iff

$$f(\lambda_1 t_1 + \lambda_2 t_2) \leqslant \lambda_1 f(t_1) + \lambda_2 f(t_2) \tag{2.1}$$

for every $t_1, t_2 \in I$, $\lambda_1, \lambda_2 \geqslant 0$ such that $\lambda_1 + \lambda_2 = 1$.

If f is a continuous function (as it will be the case in the present paper, (2.1) may be replaced by

$$f[(t_1 + t_2)/2] \leqslant [f(t_1) + f(t_2)]/2 \qquad t_1, t_2 \in I. \tag{2.2}$$

It is well known that a convex function on an interval I has lateral derivatives such that $f'(t-) \leqslant f'(t+)$ for every $t \in \text{int}(I)$. For further information on convexity, cf. HARDY, LITTLEWOOD and PÓLYA (1959) and ROBERTS and VARBERG (1973).

3. We shall say that Ω is a convex set iff

$$x, y \in \Omega \Rightarrow \lambda x + (1 - \lambda) y \in \Omega, \quad \lambda \in [0, 1]. \tag{3.1}$$

Let Ω be a convex set, $a \in \Omega$. We shall say that a is an *extreme point* of Ω iff $\Omega - \{a\}$ is still a convex set. In other words, $a \in \Omega$ is an extreme point of Ω iff

$$\begin{aligned} &a = \lambda_1 x_1 + \lambda_2 x_2 \quad \text{with} \quad \lambda_1, \lambda_2 \geqq 0, \quad \lambda_1 + \lambda_2 = 1, \quad x_1, x_2 \in \Omega \Rightarrow \\ &\Rightarrow \text{either } \lambda_1 \in \{0, 1\} \text{ or } x_1 = x_2 = a. \end{aligned} \tag{3.2}$$

4. Let Ω be a convex set in a locally convex space E (to insure the existence of sufficiently many functionals in E^* to separate points). If Ω is compact, then ex(Ω) —the set of extreme points of Ω— is necessarily non-empty (cf. PHELPS, 1966), and eventually ex(Ω) is itself a compact set. If this is the case, it is then possible to establich the following result:

Theorem 4.1 (CHOQUET, 1960). Let Ω be a convex set in a locally convex space E. If ω is compact and ex(Ω) also is compact, then any continuous linear functional L on Ω is representable by (or is the resultand of, or is the barycenter of) a regular probability measure P whose support is ex(Ω), i.e.

$$L(f) = \int_{\text{ex}(\Omega)} L(h)\, dP(h) \tag{4.1}$$

L a continuous linear functional on Ω, with $P[\Omega - \text{ex}(\Omega)] = 0$.

In case the elements of Ω are continuous functions f, observe that the evaluation functional $L_x(f) = f(x)$ is a continuous and linear one, and hence that

$$f(x) = L_x(f) = \int_{\text{ex}(\Omega)} L_x(h)\, dP(h) = \int_{\text{ex}(\Omega)} h(x)\, dP(h), \qquad f \in \Omega \tag{4.2}$$

i.e. every $f \in \Omega$ admits a Choquet type integral representation.

5. Let $\mathcal{C}$ be the set
$\mathcal{C} = \{\phi(t)\colon \phi$ is a continuous real valued real function, such that $\phi(-t) = \phi(t) \leqslant \phi(0) \in [0, 1]$, $\lim_{t\to\infty} \phi(t) = c \in [0, 1]$ and $\phi(|t|)$ is convex$\}$.

It is easily shown that $\mathcal{C}$ is a convex set; on the other hand, the evenness, continuity and convexity of ϕ imply that $\phi(|t|)$ is non-negative and non-increasing and that ϕ' exists almost everywhere, being non-positive and non-decreasing.

Let E be the locally convex space of all even, continuous, real valued real functions f, such that f' existis almost everywhere, and let us considerer the topology of uniform convergence. This topology is induced by the countable family of semi-norms

$$P_n(f) = \sup\{|f^{(i)}(|x|)|,\ n^{-1} \leqslant |x| \leqslant n,\ i = 0, 1\}$$

and hence E is metrizable; it then follows that any subset of E is compact iff it is closed and bounded in E.

Lemma 5.1. $\mathcal{C}$ is compact in E.

Proof. It is obvious that $\mathcal{C}$ is closed in E. On the other hand, a straightforward application of the mean value theorem easily shows that

$$\sup\{|f'(|x|)|,\ n^{-1} \leqslant |x| \leqslant n,\ f \in \mathcal{C}\}$$

is finite for $n = 1, 2, \ldots$, since $2/a$ is an upper bound for $-f'(|x|)$ for $x \in R -]-a, a[$, $\forall a > 0$, and hence $\mathcal{C}$ is bounded in E.

6. It is quite obvious that $\phi_0(t) \equiv 1$ and $\phi_\infty(t) \equiv 0$ are extreme points of the convex set $\mathcal{C}$ — we shall, from now on, refer to them as the degenerate extreme points of $\mathcal{C}$. On the other hand, it is straightforward to show that if ϕ is a non-degenerate extreme point of $\mathcal{C}$ then $\phi(0) = 1$ and $\lim_{t\to\infty} \phi(t) = 0$. In fact, if that was not so, we would have respectively $\phi(t) = (1 - \phi(0))\phi_\infty(t) + \phi(0)\psi(t)$ and $\phi(t) = [1 - \lim_{t\to\infty} \phi(t)]\phi_\infty(t) + \lim_{t\to\infty} \phi(t)\theta(t)$ with $\psi = [\phi(0)]^{-1}\phi$, $\theta = [\lim_{t\to\infty} \phi(t)]^{-1}\phi \in \mathcal{C}$, and this contradicts the hypothesis $\phi \in \mathrm{ex}(\mathcal{C})$. In what concerns the non-degenerate extreme points of ϕ we may further establish the results that follow:

Lemma 6.1.

The real valued real functions

$$\phi_a(t) = \max\{0, 1 - |t|/a\}, \quad a > 0 \tag{6.1}$$

are extreme points of $\mathcal{C}$.

Proof. It is obvious that $\phi_a \in \mathcal{C}$, $\forall a > 0$. Let us assume that there exist $\psi_1, \psi_2 \in \mathcal{C}$, $\lambda_1, \lambda_2 \geqslant 0$ (with $\lambda_1 + \lambda_2 = 1$) such that

$$\phi_a(t) = \lambda_1\psi_1(t) + \lambda_2\psi_2(t) \qquad \forall t \in R. \tag{6.2}$$

Differentiating we have

$$\phi_a'(t) = \lambda_1\psi_1'(t) + \lambda_2\psi_2'(t) \tag{6.3}$$

with $\phi_a'(|t|) = -a^{-1}I_{[0,a]}$ and $\psi_i'(|t|)$, $i = 1, 2$ non-positive and non-decreasing. On the other hand

$$\psi_1'(|t|) = \lambda_1^{-1}(-a^{-1} - \lambda_2\psi_2'(|t|)) \tag{6.4}$$

being the difference between a constant and a non-decreasing function is necessarily non-increasing. But if $\psi_1'(|t|)$ is simultaneously non-decreasing and non-increasing, we must conclude that $\psi_1'(|t|)$ is constant over $[0, a]$ – and the same applies to $\psi_2'(|t|)$.

Hence ψ_1, ψ_2 are linear functions over $[0, a]$. On the other hand, $\psi_1(0) = \psi_2(0) = 1$ and $\psi_1(a) = \psi_2(a) = 0$, and a is the least positive real for which ψ_1 and ψ_2 take on the value 0, and this implies that $\psi_1 \equiv \psi_2 \equiv \phi_a$— i.e. ϕ_a ex($\mathcal{C}$). (We have discarded the possibility that either $\psi_1 \equiv 0$ or $\psi_2 \equiv 0$, since these are trivial cases.)

Lemma 6.2. If ϕ is a non-degenerate extreme point of $\mathcal{C}$ then there exists an $a > 0$ such that $\phi(t) \equiv \phi_a(t) = \max\{0, 1 - |t|/a\}$.

Proof. Let ϕ be a non-degenerate extreme point of $\mathcal{C}$, $a = \inf\{x \quad R^+ : \phi(x) = 0\}$. The fact that $\phi(0) = 1$ and $\phi(|t|)$ is convex implies that there is a non-negative and non-increasing function f such that

$$\phi(|t|) = 1 - \int_0^{|t|} f(u)\,du \tag{6.5}$$

and, obviously, $f(u) = 0,\ \forall u > a$.

Let $\alpha \in\ ]0, a[$ be such that

$$1 - \int_0^{\alpha} f(u)\,du > 0 \tag{6.6}$$

and let us define

$$g(u) = \begin{cases} f(\alpha) & u < \alpha \\ f(u) & u \geqslant \alpha \end{cases} \tag{6.7}$$

and

$$\psi(|t|) = 1 - \int_0^{|t|} g(u)\,du. \tag{6.8}$$

Obviously $\phi'(|t|) - \psi'(|t|)$ is non-positive and non-decreasing, and hence $\phi - \psi = \psi^* \in \mathcal{C}$. It follows that $\phi = \psi + \psi^* \in \mathcal{C}$ and as, by hypothesis, $\phi \in \mathrm{ex}(\mathcal{C})$, this implies that

$$\psi = \lambda\phi, \qquad \lambda \in [0, 1] \tag{6.9}$$

and hence

$$\psi' = \lambda\phi' \qquad \text{a.e.,} \tag{6.10}$$

The fact that $\psi'(u) = \phi'(u)$ for $u \geqslant \alpha$ implies that $\lambda = 1$, and hence $\psi' = \phi'$ a.e. On the other hand, the fact that α had been arbitrarily chosen on $]0, a[$ implies then that $\phi'(t) = k$ (constant) over $]0, a[$, and obviously we have that $k = -a^{-1}$. Hence, finally,

$$\phi(t) = \phi_a(t) = \max\{0, 1 - |t|/a\}. \tag{6.11}$$

Lemma 6.3. The set $\mathrm{ex}(\mathcal{C})$ is compact.

PROOF. From lemmas 6.1 and 6.2 we have that

$$\mathrm{ex}(\mathcal{C}) = \{\phi_a(t),\ a \in [0, \infty]\}$$

where

$$\begin{aligned} &\phi_0(t) \equiv 1 \\ &\phi_a(t) = \max\{0, 1 - |t|/a\}, \quad 0 < a < \infty \\ &\phi_\infty(t) \equiv 0 \end{aligned} \tag{6.12}$$

Let us consider the map

$$T: [0, \infty] \to \mathcal{C} \qquad a \to \phi_a \tag{6.13}$$

Since it is not difficult to show that T is continuous, the fact that $\mathrm{ex}(\mathcal{C}) = T([0, \infty])$ is the image of a compact shows that $\mathrm{ex}(\mathcal{C})$ is itself compact.

7. In view of what we have established in the previous paragraphs the hypothesis in Choquet's theorem are satisfied for $\Omega = \mathcal{C}$. Hence every $\phi \in \mathcal{C}$ (and in particular every $\phi \in \mathcal{C}$ such that $\phi(0) = 1$) is representable by a regular probability measure P supported by $\mathrm{ex}(\mathcal{C})$, in the sense that

$$L(\phi) = \int_{\mathrm{ex}(\mathcal{C})} L(h)\, dP(h)$$

for every continuous linear functional L on $\mathcal{C}$. If in particular we consider the evaluation functional $L_t(\phi) = \phi(t)$ —which obviously is linear and continuous— we conclude that

$$\phi(t) = L_t(\phi) = \int_{\mathrm{ex}(\mathcal{C})} L_t(f)\, dP(f). \tag{7.1}$$

Let us define a measure μ over every Borel set B of $[0, \infty]$ as follows:

$$\mu(B) = P[T(B)] \tag{7.2}$$

and put $f(a) = \mu([-\infty, a])$. Now, as $L_t(\phi_a) = \max\{0, 1 - |t|/a\}$ and $T^{-1}(\mathrm{ex}(\mathcal{C})) = [0, \infty]$, we have that

$$\phi(t) = \int_{[0,\infty]} \max\{0, 1 - |t|/a\}\, dF(a) \tag{7.3}$$

or else

$$\phi(t) = \int_0^\infty \max\{0, 1 - |t|y\}\, dG(y) \tag{7.4}$$

where $G(y) = 1 - F(y^{-1})$ is the distribution function of the random variable $Y = X^{-1}$, where X has distribution function F.

Observing that $\psi(t) = \max\{0, 1 - |t|\}$ is a characteristic function corresponding to a random variable W with probability density function $f_W(x) = (2\pi)^{-1}(x/2)^{-2}\sin^2(x/2)$, we conclude that ϕ *in* (7.4) *is a characteristic function*, corresponding to the absolutely continuous random variable $Z = WY$, W and Y as described above and independent.

We have then established, as announced, the following generalized form of PÓLYA's sufficient condition for a given function to be a characteristic function:

Theorem 7.1. Let ϕ be a continuous real valued real function such that $\phi(-t) = \phi(t) \leqslant \phi(0)$ with $\phi(0) \in [0, 1]$, $\lim_{t \to \infty} \phi(t) = c \in [0, 1]$, and $\phi(|t|)$ convex over $]0, \infty[$. Then ϕ is the characteristic function of an absolutely continuous random variable.

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