TRABAJOS DE ESTADISTICA Y
DE INVESTIGACION OPERATIVA
Vol. 34, Núm. 3, 1983, pp. 95 a 108

D-OPTIMAL CYCLIC TWO-DIMENSIONAL BLOCK DESIGNS

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Abstract

In this paper we consider a class of incomplete block designs in which each block is two dimensional. Thus, hetrogeneity is removed not only between blocks, but also in two directions within each block. Such designs have been considered before in Srivastava (1977, 1978). Here, we consider the class of cyclic desings of this type when the number of treatments v is an odd number between 5 and 25, and present designs that are *D*-optimal within this class.

INTRODUCTION

Within the class of ordinary incomplete block designs, the subclass of cyclic designs is well known. (See, for example, KEMPTHORNE (1953), WOLOCK (1964), DAVID and WOLOCK (1965), etc.) Such designs have many desirable properties. This includes, for example, 'flexibility', since such designs can be made for all values of v. Ease of representation is another advantage since only one block of the design need be given, the other blocks being obtained from it in a cyclic manner. The

(*) Recibido, Septiembre, 1981

analysis of the design is not too difficult since the information matrix is a circulant, about which a rather large amount of theory is available.

However, ordinary incomplete block designs do not provide elimination of hetrogeneity in two directions. Designs that do provide for this feature, such as generalized Youden design (GYD's) of which latin squares, etc., form a subclass, suffer from the difficulty that there may be row-column interactions present. This, indeed, becomes highly likely since the number of rows and columns is usually large. In may situations, such interaction may be present even when the number of rows and columns is relatively small.

One way to handle this would be to eliminate hetrogeneity wihin each block. Thus, consider the following design with $v = 5$, and b (number of blocks) = 5, such that each block has two rows and two columns.

Table 1

1	2		2	3		3	4		4	5		5	1
3	4		4	5		5	1		1	2		2	3

Let the effect of the kth treatment ($k = 1, \ldots, v$) be denoted by τ_k, the effect of the jth block by α_j, the effect of the gth row sub-block within the jth block by ρ_{jg}, the effect of the hth column sub-clock within the jth block by γ_{jh}, and the yield in the cell (g, h) of block j by y_{jgh}. Then we shall work with the model:

(1.1a) $E(y_{jgh}) = \tau_k + \alpha_j + P_{jg} + \gamma_{jh}$,

(1.1b) Observations on distinct experimental units are independent with variance σ^2, for all permissible k, j, g, and h,

where we assume that the kth treatment is applied to the experimental unit corresponding to the cell (g, h) of block j. Notice that we are assuming an additive model for row and column effects within any block. However, obviously, this would usually be far more plausible here than in the corresponding GYD setting. This is so because in the former case, the number of rows and columns per block can be taken to be small. For example, in the design presented above, each block has only two rows and two columns. Indeed, in this paper, we shall restrict ourselves to designs of this type.

For the design presented above, it can be shown that the information matrix equals $5I_5$-J_{55}, where, throughout this paper, I_m will denote the $(m \times m)$ identity matrix, and J_{mn} the $(m \times n)$ matrix with 1 everywhere. This design is 'universally optimal,' (see KIEFER (1959)) within the class of all incomplete block designs (with one or two dimensional blocks) and with 4 replications.

Notice that in a design in which each block is of size (2×2) (i.e. has two rows and two columns) we obtain an estimate of exactly one (independent) linear function of treatments, under the above model. Thus, in
the above design, the first block gives us the estimate of $(\tau_1 + \tau_4 - - \tau_2 - \tau_3)$.
Let

(1.2) $$z_j = y_{j11} + y_{h22} - y_{j12} - y_{j21}; \quad j = 1, \ldots, b.$$

Then, clearly, $E(z_j)$ is free of the nuisance parameters $\alpha's$, $\varrho's$ and $\alpha's$. Moreover, it is easily checked that this is the only linear function of the four observations in the jth block which is free of nuisance parameters. Thus, for all practical purposes of statistical inference, we can equivalently assume that we are having observations z_j, for different values of j.

For example, for the above design, we obtain the model:

(1.3) $$\epsilon \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \\ z_5 \end{bmatrix} = \begin{bmatrix} + & - & - & + & 0 \\ 0 & + & - & - & + \\ + & 0 & + & - & - \\ - & + & 0 & + & - \\ - & - & + & 0 & + \end{bmatrix} \begin{bmatrix} \tau_1 \\ \tau_2 \\ \tau_3 \\ \tau_4 \\ \tau_5 \end{bmatrix},$$

where + and − stand for +1 and −1 respectively. Define

(1.4) $$z' = (z_1, \ldots, z_v), \quad \tau' = (\tau_1, \ldots, \tau_v).$$

Then (1.3) becomes

(1.5) $$E(z) = X\tau,$$

where the «coefficient matrix X is the (5×5) matrix on the right hand side of equation (1.3).

A general cyclic 2-dimensional block design (C2DBD) is defined as follows. We have $b = v$, and the jth block is

(1.6)
$$\begin{array}{|c|c|}\hline j+0 & j+\alpha \\ \hline j+\beta & j+\gamma \\ \hline\end{array}\,, \quad j = 1, 2, \ldots, v,$$

where α, β, γ are distinct integers satisfying $0 < \alpha, \beta, \gamma \leqslant v - 1$. Here, the addition of integers is to be done (mod v), and v is to be used in place of 0. The block corresponding to $j = v$ will be called the generating block of the design. The jth block, displayed in (1.6) may be compactly written as $((j, (j + \gamma)), ((j + \alpha), (j + \beta)))$. This block is clearly fixed by the triple (α, β, γ), knowingly which the whole design can be completely described. For the design in (1.6), the model in terms of ζ and τ is

(1.7)
$$E(\zeta) = X\tau, \; V(\zeta) = 4\sigma^2 I_v,$$

where the nature of X will be described as we proceed.

Notice that without loss of generality, we can assume that $\alpha < \beta$. Also, a design in which $\gamma < \alpha < \beta$ is identical with some design in which $\alpha < \beta < \gamma$. This is seen by noting that if the design has a block $((v, \gamma), (\alpha, \beta))$ with $\gamma < \alpha < \beta$, then it also has a block $((v - \gamma, v), (\alpha - \gamma, \beta - \gamma))$, because the latter block is obtainable by adding $(v - \gamma)$ mod (v) to each cell of the former block. The above shows that, without loss of generality, we can assume that $\alpha < \beta$, $\gamma > \alpha$, and α, β, γ are all distinct.

As an illustration, let $v = 7$, $\alpha = 1$, $\beta = 2$, and $\gamma = 4$. Then the seven blocks of the design are ((1, 5), (2, 3)), ((2, 6), (3, 4)), ((3, 7), (4, 5)), ((4, 1), (5, 6)), ((5, 2), (6, 7)), ((6, 3), (7, 1)), ((7, 4), (1, 2)).

It is clear that each block of a C2DBD provides the estimate of a linear contrast between the treatments. It follows that all rows of the X matrix add to zero, so that Rank $(X) < v - 1$. This is, of course, the case in the usual incomplete block designs. We consider designs in which the information matrix $X'X$ thus has $(v - 1)$ non-zero characteristic roots. Choosing a design with the largest value of the geometric mean of the non-zero roots then corresponds to choosing a design that is D-optimal. In this paper, we provide the values of α, β, γ for all designs which are D-optimal. This is done for all odd valued of v satisfying $5 \leqslant v \leqslant 25$. (There may exist (noncyclic) designs better than these from the viewpoint of D-optimality, but our designs are the best among those known so far.) The parameters of such optimal designs are presented in section 3, along with a computer program for generating the same for any odd value of ν. The next section presents some mathematical theory for these designs, in order to facilitate the computation needed for estimation, test of hypotheses, etc.

Unfortunately, designs of the form given by (1.6) provide only one degree of freedom for error. This should not be surprising since we have made the designs very sensitive by eliminating hetrogeneity in two directions in each block. If the aim is primarily one of estimation, the four replications provided by (1.6) should suffice. On the other hand, for purposes of hypothesis testing more degrees of freedom for error are needed unless a good prior estimate of σ^2 is available.

It is clear that $(v + 1)$ degrees of freedom for error will be available if one uses two designs of the type given in (1.6). The question is as to which two sets of (α, β, γ) should be chosen to provide the $2b$ blocks. For $v = 5$, the answer is that any two (not necesarily distinct) sets of values of (α, β, γ) will do the job. For $v = 7$, the problem has been solved by a computer search and the optimal pairs have been given in table (3). The computer program given in section 3 would enable one to find the optimal pairs for higher values of ν.

What procedure should be followed for actually using one of the designs for this paper. The first step would be to decide whether one wants one or two designs of the type (1.6). Next, if the case under consideration is not covered by table 2, one must use the computer program to generate the set (s) of values of (α, β, γ) corresponding to optimal designs. One of these designs may then be randomly chosen for actual adoption. In certain circumstances, instead of randomly choosing among the available optimal designs, one may bring in other considerations, such as practical convenience, etc. After the design has been selected, the treatments, blocks, and the sub blocks should be randomized as usual.

We close this section by providing some examples of situations where such designs would be extremely useful. The first one concerns agricultural experiments on mountain terraces. Often, the terraces are not too big. Indeed, they may often not be able to hold more than one plot (of a size that would normally be used on an experimental farm in the plains). On these terraces, very wide variations in soil and micro-climatic conditions can be expected. Thus, the conventional block designs will not be advisable. However, designs in which each block is two dimensional (preferably of size (2×2) as in the present paper) would be called for. Four adjacent terraces could be used to form a block, these being selected such that they form two adjacent pairs, each pair consisting of a terrace at one level and a terrace just below it at the next lower level. The two pairs so formed may be considered to correspond to the two column sub-blocks. Similarly, the two terraces at the higher level, versus the two at the lower level, would correspond to row sub-blocks.

The next example which we present is from the field of education, though it could be easily adapted to other fields such as marketing. Suppose we wish to compare ν methods of audio-visual instruction. We may choose v towns, spread over the area of investigation. In each town we may randomly select two schools and two teachers (not from the two selected schools). The towns then constitute the blocks. The schools and the teachers may be considered to correspond to row and column subblocks. Each teacher will provide instruction in each of the two schools according to the selected design. Notice that the design will effectively eliminate the variation between schools, which may be large, and similarly, the variation between the teachers. The towns will provide a good coverage, and yet the differences between them will be eliminated. Notice that the ordinary type of incomplete block designs on GYD's are simply not applicable since the two schools or the two teachers obviously do not form a factor cross-classified with the town.

The last example concerns the comparison of v varieties of a crop over a large territory, such as a county. One may select b counties, and two farms in each county. Another variable of classification may be selected in each county. This variable may differ from county to county, and may correspond to various important agricultural practices being followed. For example, in a certain county there may be a significant proportion of irrigated and of non-irrigated lands on which crops are grown. In this case, irrigation could be the variable under consideration. On the other hand, in another county two different competing methods of cultivation might be in vogue. Here, the variable under consideration would correspond to the method of cultivation.

Notice that we are able to use different variables for defining sub-blocks in different counties (blocks), because our designs do not require cross classifcation of factors. It is apparent that an experiment of this kind would not only be sensitive towards varietal comparisons, but would also provide for a large base. In other words, the varietal comparisons from an experiment of the recommended type will give us a better picture of how the varieties stand in comparison to each other, not only at the farm of some experiment station, but in the territory at large, reflecting the various farming conditions and practices being followed.

2. THE COVARIANCE MATRIX

Circulant matrices play an important role in the theory of cyclic designs. Below we present a few known results concerning these which are then used to obtain the covariance matrix of the estimates. Using this, statistical inference problems can be easily tackled.

A circulant is a matrix C of the form

$$C = \begin{bmatrix} c_0 & c_{v-1} & \dots & c_1 \\ c_1 & c_0 & \dots & c_0 \\ c_{v-1} & c_{v-2} & \dots & c_0 \end{bmatrix}. \tag{2.1}$$

Let Q be obtained from C by taking $c_1 = 1$ and the other c's equal to zero. Then it can be easily checked that

$$C = \sum_{j=0}^{v-1} c_j Q^j, \; Q^v = I_v \tag{2.2}$$

Let D be a matrix obtained from C by replacing the c_i by d_i for all i. Then we get

$$CD = \sum_{u=0}^{v-1} f_u Q^u, \quad \text{where} \tag{2.3}$$

$$f_u \; = \sum_{\substack{s,t=0 \\ s+t=u(\text{mod } v)}}^{v-1} c_s d_t; \quad u = 0, \dots, v-1. \tag{2.4}$$

Thus, the circulants form a linear associative algebra, say ζ. Hence, if C is non-singular, then $C^{-1} \epsilon \zeta$, and hence C^{-1} is also a circulant. Also, since $Q' = Q^{v-1}$, it follows that

(2.5a) $$C' = \sum_{j=0}^{v-1} c_{v-j} Q^j, \quad \text{with}$$

(2.5b) $$c_v \equiv c_0.$$

Thus, C', CC', and $C'C$ all are circulants. Now, let $\omega = e^{2\pi i/v}$, where $i = \sqrt{-1}$, so that ψ is a νth root of unity. Also, let

(2.6) $$\epsilon_s' = \frac{1}{\sqrt{v}}(\omega_s, \omega_s^2, \ldots, \omega_s^v); \quad s = 1, \ldots, v; \quad \text{where}$$

(2.7) $$\omega_s = \omega^{v-s} = e^{2\pi i(5v-s)/v}.$$

As is customary, for any matrix Ω, we shall denote the conjugate-transpose of Ω by Ω^*. Then, using the fact that the conjugate of $e^{i\theta}$ is $e^{-i\theta}$, it is easy to check that

(2.8) $$\epsilon_s^* \epsilon_s = 1; \quad \epsilon_s^* \epsilon_t = 0; \quad s, t = 1, \ldots, v.$$

Define

(2.9) $$\phi(\mu) = c_0 + c_1\mu + c_2\mu^2 + \ldots + c_{v-1}\mu^{v-1},$$

(2.10) $$E = [\epsilon_1, \epsilon_2, \ldots, \epsilon_v]$$

Then, we can verify that

(2.11) $$C\epsilon_s = (\phi(\omega^s))\epsilon_s; \quad s = 1, \ldots, v,$$

(2.12) $$E^*CE = \text{diag}[(\phi(\omega)), (\phi(\omega^2)), \ldots, (\phi(\omega^v))]$$

Thus, the characteristic roots of C are $\phi(\omega^k)$, for $k = 1, \ldots, v$.

Recall the model (1.5). As usual, we take $\sum_{k=1}^{v} \tau_k = 0$, or equivalently,

(2.13) $$J_{vv}\tau = 0_{vl},$$

where 0_{mn} denotes a zero matrix of size $(m \times n)$. Then, the model for z becomes

(2.14) $$E(z) = (X + J)\tau, \; V(z) = 4\sigma^2 I_v.$$

Theorem. 2.1 Let $M(v \times v)$ be a symmetric matrix of rank $(v - 1)$, such that all its rows and columns sum to zero. Then the following hold:

(i) $(M + J)$ is non-singular,

(ii) $\frac{1}{v}|M + J|$ = Product of nonzero roots of M,

(iii) Let $R((v - 1) \times v)$ be a matrix such that $\left[\frac{R}{1/\sqrt{v}J_{1v}}\right]$ is orthogonal. Then $|[RM^{-}R']|$ equals the reciprocal of the nonzero roots of M, where $(-)$ in the superscript of M^{-} denotes a 'generalized' or 'conditional' inverse.

(iv) If M corresponds to the information matrix of a design, then a design which maximizes the product of the nonzero roots of M is D-optimal.

PROOF. Since $MJ_{vv} = J_{vv}M = 0_{vv}$, there exists an orthogonal matrix L such that $L'ML = \text{diag}(m_1, \ldots, m_{v-1}, 0)$, and $L'JL = \text{diag}(0, \ldots, 0, v)$, where the m's are nonzero. Results (i) and (ii) follow from this. Also, clearly, $L = [L_1 \vdots \frac{1}{\sqrt{v}} J_{vl}]$, where the columns of L_1 are orthogonal to J_{vi}. Since $M = L\{\text{diag}(m_1, \ldots, m_{v-1}, 0)\}L'$, *every* generalized inverse M' of M is of the form $M^{-} = L\{\text{diag}(m_1^{-1}, \ldots, m_{v-1}^{-1}, m_0)\}L'$, where m_0 is some real number. Hence,

$$|RM^{-1}R'| = \left|R\left[L_1 \vdots \frac{1}{\sqrt{v}} J_{vl}\right]\{\text{dig}(m_1^{-1}, \ldots, m_{v-1}^{-1}, m_0)\}\begin{bmatrix} L_1' \\ \frac{1}{\sqrt{v}} J_{1v}\end{bmatrix} R'\right|$$

$$= |RL_1\{\text{diag}(m_1^{-1}, \ldots, m_{v-1}^{-1})\}L_1'R'|$$

$$= |RL_1|^2(m_1 \cdot m_2 \cdots m_{v-1})^{-1}$$

But

$$1 = \left|\begin{bmatrix} R \\ \frac{1}{\sqrt{v}} J_{1v}\end{bmatrix}\left[L_1 \vdots \frac{1}{\sqrt{v}} J_{v1}\right]\right|^2 = \begin{vmatrix} RL_1 & 0 \\ 0 & 1\end{vmatrix}^2 = |RL_1|^2.$$

This proves (iii).

The result (iv) follows from (iii) and the definition of D-optimality.

Theorem 2.2 Let X be as in (1.5). Then the following results hold:

(i) X equals the circulant c in (1.1), with $c_0 = c_{v-\gamma} = 1$, $c_{v-\beta} = -1$, and the other $c's$ being zero.

(ii) If rank $X = v - 1$, then the product of the nonzero roots of X equals $1/v|X + J|$, and the product of the nonzero roots of $X'X$ equals $1/v^2|X + J|^2$.

$$\text{Let } \psi(\mu) = 1 - \mu^{-\alpha} - \mu^{-\beta} + \mu^{-\gamma}. \tag{2.15}$$

Then the roots of X are $\psi(\omega^k)$, $k = 1, \ldots, v$.

(iv) Given v, and $b = v$, in order to obtain a D-optimal C2BDB, we must maximize

$$\Delta = \left[\prod_{k=1}^{v-1} (1 - e^{-2\pi i k\alpha/v} - e^{-2\pi i k\beta/v} + e^{-2\pi i k\gamma/v}) \right] \tag{2.16}$$

with respect to the parameters (α, β, γ).

Proof. Statement (i) is obvious. Since every row (and column of X has two $(+1)$'s and two (-1)'s, it is clear that

$$XJ_{vv} = J_{vv}X = 0_{vv}. \tag{2.17}$$

Hence, there exists a matrix W such that $W^*XW = D_1$, and $W^*J_{vv}W = D_2$, where D_1 and D_2 are lower triangular matrices, which contain the roots of X and J respectively on their diagonals. Since the only nonzero root of J is v, suppose that the (1, 1) element of D_2 is v, and other diagonal elements of D_2 are zero. Then, because of (2.17), the (1, 1) element of D_1 must be zero, while the other diagonal elements are nonzero (since rank $(X) = v - 1$). Thus, $|D_1 + D_2|$ equals v times the products of the nonzero roots of X. But $|D_1 + D_2| = |W(D_1 + D_2)W^*| = |X + J|$. This proves the first part of (ii). By similar reasoning, and using the fact that XX' and J commute, it follows that $|X'X + vJ| = v^2$ (product of the roots of $X'X$). The proof of (ii) is completed by observing that $|X' + J||X + J| = |XX' + vJ|$. Part (iii) follows from (i) and the remark after (2.12). Part (iv) follows from (iii) and part (iv) of theorem (2.1). Throughout this paper, for any matrix M, the jth column will be denoted by $(M)_j$. Also, let $(\alpha', \beta', \gamma')$ be the ordered form of the triplet (α, β, γ), such that $0 < \alpha' < \beta' < \gamma' \leqslant v - 1$.

Theorem 2.3 Let X be as in (1.5). The, the following hold:

(i) $X'X$ is a circulant. If rank $(X) = v - 1$, then $(X'X + J_{vv})^{-1}$ is a conditional inverse of $X'X$, and is a circulant.

(ii) Out of the 12 numbers α', β', γ', $(\beta' - \alpha')$, $(\gamma' - \alpha')$, $(\gamma' - \beta')$, $(v - \alpha')$, $(v - \beta')$, $(v - \gamma')$, $(v - \beta' + \alpha')$, $(v - \gamma' + \alpha')$, and $(v - \gamma' + \beta')$, suppose exactly n are distinct. Let $\theta_1 < \theta_2 < \cdots < \theta_n$ be these numbers. Then the nonzero elements of $(X'X)$, can occur only at the coordinates numbered $(1 + \theta_{n'})$, where $n' = 0, 1, \ldots, n$, and where $\theta_0 = 0$.

PROOF. Part (i) follows from the remarks made before and after (2.5), and using an argument similar to that in the proof of the last two theorems. To prove part (ii), first observe that the $(k + 1)$th column of X is obtained from the kth column by a one step cyclic shift, which consists of placing the last element of the kth column at the top. Now, the successive elements of $(X'X)$, are obtained by taking the scalar product of $(X)_1$ with $(X)_k$, for $k = 1, \ldots, v$. The column (X), and hence $(X)_k$, for all k, have only 4 nonzero elements, which in $(X)_i$ matches against a nonzero element $(X)_k$. This will happen, for example, when $k = 1 + (1 + \beta') - (1 + \alpha')$. The reason is that the column $(X)_{1+\beta'-\alpha'}$ is obtained by shifting the elements of $(X)_1$ by $(\beta' - \alpha')$ places, so that the element in position $(1 + \alpha')$ in $(X)_1$ stands at the position $(1 + \beta')$ in $(X)_{1+\beta'-\alpha'}$. Similarly, since the column $(X)_{1+v-\beta'+\alpha')}$ is obtained by shifting the elements of $(X)_1$ by $(v - \beta' + \alpha')$ places, the element in the position $(1 + \beta')$ in $(X)_1$ stands at the position $(1 + \alpha')$ in $(X)_{1+v-\beta'+\alpha'}$. Thus, in the 4 plet $(1, 1 + \alpha', 1 + \beta', 1 + \gamma')$, each pair gives rise to possibly two columns, which could give a nonzero element when multiplied with $(X)_1$. This completes the proof.

Theorem 2.4

(i) If C at (2.1) is symmetric and v is odd, then the characteristic roots of C are given by $\zeta_s (s = 1, \ldots, v)$, where

$$\zeta_s = c_0 + 2 \sum_{k=1}^{(v-1)/2} c_k \cos(2\pi sk/v). \tag{2.18}$$

(Notice that $\zeta_s = \zeta_{v-s}$, for all permissible s.)

(ii) If, furthermore, C is nonsingular, then $(C^{-1})_{j+1,1}$, the $(j + 1)$th element ($j = 0, 1, 2, \ldots, v - 1$) in the first column of C^{-1} (which is a circulant) is given by

$$(2.19) \qquad (C^{-1})_{j+1,1} = \frac{2}{v} \sum_{s=1}^{(v-1)/2} \zeta_s^{-1} \cos(2\pi sj/v).$$

PROOF

(i) Since C is symmetric, we have $c_k = c_{v-k}$, $k = 1, \ldots, (v-1)/2$. Hence, recalling the remark after (2.12), we find that the root $\phi(\omega^s)$ becomes $c_0 + \sum_{k=0}^{v} c_k e^{2\pi isk/v}$, which equals (2.18), by DeMoivre's Theorem.

(ii) From (2.12), we have

$$(2.20) \qquad C^{-1} = E\{\operatorname{diag}([\phi(\omega)]^{-1}, \ldots, [\phi(\omega^v)]^{-1})\}E^*.$$

$$= \sum_{k=1}^{v} [\phi(\omega^k)]^{-1} \epsilon_k \epsilon_k^*.$$

Hence, (2.6) and (2.7) give

$$(C^{-1})_{j+1,1} = \frac{1}{v} \sum_{k=1}^{v} [\phi(\omega^k)]^{-1} e^{2\pi i(v-k)j/v},$$

which leads to (2.19). This completes the proof.

Theorem (2.1) tells us how to deal with a matrix of the type M (which is the kind that arises in this paper). Part (iv) of this theorem, in conjunction with part (ii) of theorem (2.2) lead us to the formulae used in the computer program. Theorem (2.3) enables one to quickly compute $X'X$. Indeed, using it, one can compute within a few minutes the nonzero elements in the first column of $X'X$, and the positions at which they occur. Substituting this information in (2.18) and (2.19), leads us to the covariance matrix, (Here, we have to be reminded that part (i) of theorem (2.3) should be used.) Thus, the procedure for computing $X'X$, and a conditional inverse of the same is quite simple. Given the design, using the triplet (α, β, γ), we first determine $(\alpha', \beta', \gamma')$, and $(X'X)_1$. If more than one design is used, we determine $(X'X)_1$ for each design, and then add these to obtain $(M)_1$, where M denotes the information matrix of the composite design. Then by using

(2.18) and (2.19), we can obtain a conditional inverse of M, say M^-. The estimate $\hat{\tau}$ of τ is then given by

(2.21) $\hat{\tau} = M^{-1}[X^{(1)\prime}z^{(1)} + \ldots + X^{(d)\prime}z^{(d)}]$, where d designs are being used (giving rise to $4d$ replications), where $X^{(q)}$ is the $(v \times v)$ X matrix corresponding to the qth design, and where $Z^{(q)}$ is the $(v \times 1)$ vector of 'observations' from the qth design. Also,

$$\text{var}(\hat{\tau}) = 4\sigma^2 M^-, \tag{2.22}$$

which can be used in the usual hypothesis testing & confindence-interval formulae.

3. TABLES OF DESIGNS AND COMPUTER PROGRAMS

In this section, we present the parameters (α, β, γ) for D-optimal designs with odd values of v between 7 and 25. As mentioned earlier, for all values of v, except $v = 7$, we present designs providing 4 replications. For $v = 7$, we provide designs with 8 replications, and thus present the two pairs of triplets (α, β, γ) for each optimal design. For each value of v, we present the value of Δ which equals the geometric mean of the nonzero roots of the information matrix, and also of Δ^{v-1} (which is an integer). Also N_v denotes the total number of optimal designs, and hence, the total number of sets of parameters presented.

Table (2) give the designs with 4 replications and Table (3) with 8 replications.

Computer program is written in Fortran 5 and has been used in the CDC 120-720 computer al Colorado State University. Furthermore, information will be found in the comments given along with the program.

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This research was supported by AFOSR Grant #77-3127

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