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SIMPLE LARGE SAMPLE ESTIMATORS OF SCALE AND LOCATION PARAMETERS BASED ON BLOCKS OF ORDER STATISTICS

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ABSTRACT

In this paper quite efficient large sample estimation procedures are derived for jointly estimating the parameters of the location-scale family of distributions. These estimators are linear combinations of the means of suitably chosen blocks of order statistics. For specific distributions, such as the extreme-value, normal, and logistic, little is to be gained by using more than three blocks. For these distributions we can obtain joint relative asymptotic efficiencies of 97-98% using the means of three blocks of ordered observations. The estimation procedures are also adapted for the estimation of the shape and scale parameters of the Weibull distribution.

1. Introduction

During recent years a large amount of effort has been devoted to finding estimators of scale (δ) and location (λ) parameters based on linear functions of order statistics. Optimal asymptotic estimation by order statistics, introduced by Mosteller [11] and Bennett [4], was followed by numerous authors and considerably extended by Chernoff et al. [5].

In the present paper we examine a class of large sample estimators for the scale and location parameters which are based on the sample means of two or three suitably chosen blocks of ordered observations.

More specifically, for a location-scale family of continuous distributions, $F((x - \lambda)/\delta)$, we find estimators of λ and δ which are linear combinations of the means of the observations in the left and right tails of the sample, i.e.,

$$X_N^* = \frac{1}{N} \sum_{i=1}^{N} X_{(i)}$$

$$X_M^{**} = \frac{1}{n-M} \sum_{i=M+1}^{n} X_{(i)}$$

where $X_{(1)} \leqslant \cdots \leqslant X_{(N)} \leqslant \cdots \leqslant X_{(M)} \leqslant \cdots \leqslant X_{(n)}$ denote the order statistics associated with a random sample of size n from the distribution $F((x - \lambda)/\delta)$.

Furthermore, we give estimators of λ and δ using X_N^* and X_M^{**} and the sample mean of a middle portion of the ordered observations, i.e.

$$\tilde{X}_{M,N} = \frac{1}{M-N} \sum_{i=N+1}^{M} X_{(i)}$$

We focus our attention on estimators which are asymptotically unbiased and possess the smallest variance within the class studied. At least for the distributions considered here, these estimators are shown to be quite efficient and very simple to calculate.

Only complete samples will be considered but, in principle, the estimation procedures can be easily adjusted for any kind of censoring [10].

The importance and practical application of the tails in estimation problems were already mentioned by Mosteller and Tukey [12]. For finite samples, Jones [9] used the statistic

$$S'_n = \sum_{j=n-N+1}^{n} X_{(j)} - \sum_{j=1}^{N} X_{(j)}, \quad N < \frac{n}{2}$$

to estimate the standard deviation σ of the normal distribution and derived the expressions for $E(S'_n)$ and Var (S'_n).

D'Agostino and Cureton [7] used Jones' idea and proposed an asymptotically unbiased estimator of σ, the standard deviation of the normal distribution of the form:

$$\sigma_p = \frac{S'_n}{n} \cdot d_p, \quad p = \frac{N}{n}$$

which has been found to be very efficient and fairly robust.

For doubly censored samples Abe [1] investigated a class of estimators of the scale and location parameters based only on sample tails, permitting extra weights on the smallest and largest known observation. The performance of these estimators in the normal distribution was studied in [2]. For complete samples our paper extends the results of [1] and also suggests the estimators in cases where some middle part of observations is censored.

In the next section, we introduce briefly the asymptotic distribution theory for the sample tail and block means. Most of the proofs are omitted, since the asymptotic normality is not surprising and the normalizing constants can be derived using, for instance, the results of [5]. More detailed treatment and proofs can be found in [10].

In sections 3 and 4 we derive the estimators and obtain their asymptotic efficiencies. Finally, in section 5 the estimation procedure is used to estimate the parameters of the Gumbel, normal, logistic, and Weibull distributions.

2. Asymptotic distribution theory for tail means

Let $U_1, \ldots, U_n$ be i.i.d uniform (0, 1) random variables and let

$U_{(1)} \leqslant U_{(2)} \leqslant \cdots \leqslant U_{(n)}$ denote their order statistics, arranged in order of increasing size. Define the statistic U_N^*

$$U_N^* = \frac{1}{N} \sum_{j=1}^{N} g(U_{(j)}) \tag{2.1}$$

where $N = [np]$ denotes the largest integer not exceeding np, $0 < p < 1$, and $g(\cdot)$ is some real function defined over the interval $(0, p]$.

Assumption A: $g(\cdot)$ is third order differentiable (*a. e.*) over the interval $(0, p]$, $0 < p < 1$ and the integral

$$\int_0^p g^2(w)\, dw \quad \text{is finite}$$

Theorem 1: If assumption A is satisfied then

$$\frac{U_N^* - \mu_1(p)}{\sigma_1(p)} \sqrt{np}$$

is asymptotically distributed $N(0, 1)$. The normalizing constants are defined as

$$\mu_1(p) = \frac{1}{p} \int_0^p g(w)\, dw \tag{2.2}$$

and

$$\sigma_1^2(p) = \frac{1}{p} \int_0^p g^2(w)\, dw - \mu_1^2(p) + [g(p) - \mu_1(p)]^2\, (1 - p) \tag{2.3}$$

Proof: follows from Corollary 3 and 4 of [5]. For a different proof see also [10].

Similarly we define U_M^{**}

$$U_M^{**} = \frac{1}{n - M} \sum_{j=M+1}^{n} g(U_{(j)}) \quad , \tag{2.4}$$

where $M = [nq]$. Let us assume that $g(\cdot)$ is defined on $(0, p] \cup [q, 1)$ and $0 < p \leqslant q < 1$.

Assumption B: $g(\cdot)$ is third order differentiable (a.e.) on the union of intervals $(0, p] \cup [q, 1)$ and the integrals

$$\int_0^p g^2(w)\, dw \quad \text{and} \quad \int_q^1 g^2(w)\, dw \quad \text{are finite}$$

With the notation

$$\mu_2(q) = \frac{1}{1-q} \int_q^1 g(w)\, dw \quad , \tag{2.5}$$

$$\sigma_2^2(q) = \int_q^1 g^2(w) \frac{dw}{1-q} - \mu_2^2(q) + [\mu_2(q) - g(q)]^2\, q \tag{2.6}$$

and

$$\sigma_{12}(p, q) = [\mu_2(q) - g(q)] \cdot [g(p) - \mu_1(p)] \tag{2.7}$$

we state the bivariate version of the previous theorem.

Theorem 2: If assumption B is satisfied then

$$\frac{U_N^* - \mu_1(p)}{\sigma_1(p)} \sqrt{np} \quad \text{and} \quad \frac{U_M^{**} - \mu_2(q)}{\sigma_2(q)} \sqrt{n(1-q)}$$

are jointly asymptotically distributed

with bivariate normal distribution

$$N\left(\begin{bmatrix}0\\0\end{bmatrix}, \begin{bmatrix}1 & \rho\\ \rho & 1\end{bmatrix}\right)$$

Notation: Here ρ denotes the coefficient of correlation, and is given by

$$\rho = \frac{\sigma_{12}(p, q)}{\sigma_1(p)\,\sigma_2(q)} \sqrt{p\,(1-q)} \tag{2.8}$$

The normalizing constants $\mu_1(p)$ and $\sigma_1^2(p)$ are as previously defined in Theorem 1.

Finally, we consider $\tilde{U}_{N,M}$ defined by

$$\tilde{U}_{N,M} = \frac{1}{M-N} \sum_{j=N+1}^{M} g(U_{(j)}) \tag{2.9}$$

We note that the statistic $\tilde{U}_{N,M}$ appears in slightly different forms in much of the literature on robust estimation of location (see e.g., David [8], Andrews et al. [3]).

Assumption C: With $0 < p < q < 1$, the function $g(\cdot)$ is third-order differentiable over the interval $[p, q]$ and $\int_p^q g^2(w)\,dw$ is finite.

Theorem 3: If assumption C is satisfied:

$$\frac{\tilde{U}_{N,M} - \mu_3(p, q)}{\sigma_3(p, q)} \sqrt{n\,(q-p)}$$

is asymptotically distributed $N(0, 1)$, with the normalizing constants

$$\mu_3(p, q) = \frac{1}{q-p} \int_p^q g(w)\,dw \tag{2.10}$$

and

$$\sigma_3^2(p, q) = \int_p^q g^2(w) \frac{dw}{q-p} - [\mu_3(p, q)]^2 +$$

$$+ \frac{1}{q-p} \{p(1-p)[\mu_3(p, q) - g(p)]^2 + q(1-q) \cdot \qquad (2.11)$$

$$\cdot [g(q) - \mu_3(p, q)]^2 + 2p(1-q) \cdot [\mu_3(p, q) - g(p)] \cdot [g(q) - \mu_3(p, q)]\}$$

The next two theorems state the joint asymptotic normality of a sample tail mean and the sample mean of an adjacent block of observations.

Theorem 4: Suppose that Assumptions A and C are satisfied. Then U_N^* and $\tilde{U}_{N,M}$ are jointly asymptotically normally distributed

$$N\left[\begin{bmatrix} \mu_1(p) \\ \\ \mu_3(p, q) \end{bmatrix}, \frac{1}{n}\begin{bmatrix} \frac{\sigma_1^2(p)}{p} & \sigma_{13}(p, q) \\ \\ \sigma_{13}(p, q) & \frac{\sigma_3^2(p, q)}{q-p} \end{bmatrix}\right],$$

where

$$\sigma_{13}(p, q) = (g(p) - \mu_1(p)) \cdot$$

$$\cdot \left[\mu_3(p, q) + \frac{1-q}{q-p} g(q) - \frac{1-p}{q-p} g(p)\right] \qquad (2.12)$$

with $\mu_1(p)$, $\mu_3(p, q)$, $\sigma_1^2(p)$, $\sigma_3^2(p, q)$ defined by (2.2), (2.10), (2.3), (2.11) respectively.

Theorem 5: Under the assumptions B and C, $\tilde{U}_{N,M}$ and U_M^{**} are jointly asymptotically normally distributed with the covariance term

$$\sigma_{23}(p, q) = (\mu_2(q) - g(q)) \cdot$$

$$\cdot\left[-\mu_3(p,\ q)+\frac{q}{q-p}\,g(q)-\frac{p}{q-p}\,g(p)\right] \qquad (2.13)$$

and with $\mu_3(p,\ q)$, $\mu_2(q)$, $\sigma_3^2(p,\ q)$ and $\sigma_2^2(q)$ defined as before.

We note that in Theorem 2 we made an assumption that $0 < p \leqslant \leqslant q < 1$. In this case the left and right tails have no point in common.

In the other case, when $1 > p > q > 0$, both tails of the sample overlap and both contain the $N - M$ middle observations (100 $(p - q)$ percent of the sample):

$$U_{(1)} \leqslant \cdots \leqslant \underbrace{U_{(M)} \leqslant \overbrace{U_{(M+1)} \leqslant \cdots \leqslant U_{(N)}}^{\text{right tail}}}_{\text{left tail}} \leqslant U_{(N+1)} \leqslant \cdots \leqslant U_{(n)}$$

For the second case Theorem 2 remains valid, provided that $\sigma_{12}(p,\ q)$, formerly given by (2.7), is now expressed as follows:

$$\begin{aligned}\sigma_{12}(p,\ q)=\frac{1}{p\,(1-q)}\Bigg\{&\int_q^p g^2(w)\,dw-(p-q)\,\mu_3(q,\ p)+\\ &+q\,(1-p)\,(\mu_2(p)-g(p))\,(g(q)-\mu_1(q))+\\ &+q\,(\mu_3(q,\ p)-g(q))\,(\mu_2(p)-\mu_1(q))+ \qquad (2.14)\\ &+[q\,(\mu_3(q,\ p)-\mu_1(q))+p\,(\mu_2(p)-\mu_3(q,\ p))]\cdot\\ &\cdot[(1-p)\,(g(p)-\mu_3(q,\ p))-q\,(\mu_3(q,\ p)-g(q))]\Bigg\}\end{aligned}$$

Derivation of (2.14): We can write

$$U_N^* = \frac{1}{N}\sum_{j=1}^{N} g(U_{(j)}) = \frac{1}{N}\left[\frac{M}{M}\sum_{j=1}^{M} g(U_{(j)}) + \frac{N-M}{N-M}\sum_{j=M+1}^{N} g(U_{(j)})\right] =$$

$$= \frac{M}{N} U_M^* + \frac{N - M}{N} \tilde{U}_{M,N} \tag{2.15}$$

and similarly

$$U_M^{**} = \frac{1}{n - M} \sum_{j=M+1}^{n} g(U_{(j)}) = \frac{N - M}{n - M} \tilde{U}_{M,N} + \frac{n - N}{n - M} U_N^{**} \quad \cdot \tag{2.16}$$

Define $\xi = \lim_{n \to \infty} (M/N)$ and $\gamma = \lim_{n \to \infty} (n - N)/(n - M)$.

Clearly $\xi = q/p$, $\gamma = (1 - p)/(1 - q)$ and thus

$$U_N^* \xrightarrow{p} \xi \mu_1 + (1 - \xi) \mu_3, \quad U_M^{**} \xrightarrow{p} \gamma \mu_2 + (1 - \gamma) \mu_3, \quad \text{as} \quad n \to \infty \quad ,$$

where, for brevity, we use the notation $\mu_1 = \mu_1(q)$, $\mu_2 = \mu_2(p)$ and $\mu_3 = \mu_3(q, p)$.

We calculate

$$\frac{\sigma_{12}(p, q)}{n} \simeq^{\dagger} \text{Cov}\,(U_N^*, U_M^{**}) \simeq \xi \gamma \,\text{Cov}\,(U_M^*, U_N^{**}) +$$

$$+ (1 - \xi) \gamma \text{Cov}\,(\tilde{U}_{M,N}, U_N^{**}) + \xi (1 - \gamma) \,\text{Cov}(U_M^*, \tilde{U}_{M,N}) +$$

$$+ (1 - \gamma)(1 - \xi) \,\text{Var}\,(\tilde{U}_{M,N}) \tag{2.17}$$

then replace the variance - covariance terms in (2.17) and after some rearrangements and calculations (2.15) follows easily.

We also state a bivariate version of Theorem 3. Consider

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† $\simeq$ ... asymptotically equal, in fact we have

$$\lim_{n \to \infty} n \,\text{Cov}(U_N^*, U_M^{**}) = \sigma_{12}(p, q)$$

$$\tilde{U}_{N_1, N_2} = \frac{1}{N_2 - N_1} \sum_{j=N_1+1}^{N_2} g(U_{(j)}) \tag{2.18}$$

and

$$\tilde{U}_{M_1, M_2} = \frac{1}{M_2 - M_1} \sum_{j=M_1+1}^{M_2} g(U_{(j)}) \tag{2.19}$$

where $N_i = [np_i]$, $M_i = [nq_i]$, $i = 1, 2$, and $0 < p_1 < p_2 < q_1 < q_2 < 1$.

Theorem 6: If the conditions of Theorem 3 hold for the intervals $[p_1, p_2]$ and $[q_1, q_2]$ then $\tilde{U}_{N_1, N_2}$ and $\tilde{U}_{M_1, M_2}$ are jointly asymptotically normally distributed

$$N\left[\begin{bmatrix} \mu_3(p_1, p_2) \\ \\ \mu_3(q_1, q_2) \end{bmatrix}; \frac{1}{n}\begin{bmatrix} \dfrac{\sigma_3^2(p_1, p_2)}{p_2 - p_1} & \sigma_{12}(p_1, p_2; q_1, q_2) \\ \\ \sigma_{12}(p_1, p_2; q_1, q_2) & \dfrac{\sigma_3^2(q_1, q_2)}{q_2 - q_1} \end{bmatrix}\right]$$

where

$$\begin{aligned} \sigma_{12}(p_1, p_2; q_1, q_2) = {} & [(q_2 - q_1)(p_2 - p_1)]^{-1} \cdot \\ & \cdot [p_2(g(p_2) - \mu_3(p_1, p_2)) - \\ & - p_1(g(p_1) - \mu_3(p_1, p_2))] \cdot \\ & \cdot [(1 - q_2)(g(q_2) - \mu_3(q_1, q_2)) - \\ & - (1 - q_1)(g(q_1) - \mu_3(q_1, q_2))] \end{aligned} \tag{2.20}$$

The expectations $\mu_3(\cdot,\cdot)$ and the variance terms $\sigma_3^2(\cdot,\cdot)$ are given by (2.10) and (2.11) respectively.

3. Choice of estimators

Let $X_1, X_2, \ldots, X_n$ be a random sample from a population possessing the continuous c.d.f. $F\left(\frac{x-\lambda}{\delta}\right)$ and density $\delta^{-1} f\left(\frac{x-\lambda}{\delta}\right)$. Denote by $X_{(1)} \leqslant X_{(2)} \leqslant \cdots \leqslant X_{(n)}$, the order statistics corresponding to the above sample. The parameters λ and δ are both unknown.

(a) First, we shall consider the estimators of λ and δ based only on sample tail means X_N^* and X_M^{**} defined by

$$X_N^* = \frac{1}{N} \sum_{j=1}^{N} X_{(j)} \tag{3.1}$$

and

$$X_M^{**} = \frac{1}{n-M} \sum_{j=M+1}^{n} X_{(j)} \tag{3.2}$$

where $N = [np]$, $M = [nq]$ with $0 < p, q < 1$ fixed.

The standardized random variables Z_i, $i = 1, \ldots, n$ which are related to the X_i, $i = 1, \ldots, n$ by $Z_i = (X_i - \lambda)/\delta$, $i = 1, \ldots, n$, are i.i.d. random variables distributed with c.d.f. $F(x)$ and p.d.f. $f(x)$. Note that $F(x)$ and $f(x)$ are parameter-free and hence completely known.

Denoting by Z_N^* and Z_M^{**} the sample tail means of the standardized variable Z one may write

$$X_N^* = \lambda + \delta Z_N^*$$
$$X_M^{**} = \lambda + \delta Z_M^{**}$$

By Theorem 2, $Z_N^* \xrightarrow{p} \mu_1(p)$ and $Z_M^{**} \xrightarrow{p} \mu_2(q)$ as $n \to \infty$, where $\mu_1(p)$ and $\mu_2(q)$ are:

$$\mu_1(p) = \frac{1}{p} \int_{-\infty}^{x_p} x \, dF(x) \tag{3.3}$$

$$\mu_2(q) = \frac{1}{1-q} \int_{x_q}^{\infty} x \, dF(x) \tag{3.4}$$

where x_p and x_q are the p and q quantiles of the distribution $F(x)$ defined by equations $x_p = F^{-1}(p)$ and $x_q = F^{-1}(q)$. Clearly, in our case we have: $g(w) \equiv F^{-1}(w)$, where $F^{-1}(\cdot)$ denotes the inverse of $F(\cdot)$, and asymptotic normality is evident, provided that $F^{-1}(w)$ satisfied the assumption of Theorem 2.

Therefore, as $n \to \infty$, we obtain:

$$X_N^* \xrightarrow{p} \lambda + \delta \, \mu_1(p)$$

and

$$X_M^{**} \xrightarrow{p} \lambda + \delta \, \mu_2(q)$$

Then the natural way to define the asymptotically unbiased estimators for λ and δ is

$$\lambda_n^* = \frac{\mu_2(q) X_N^* - \mu_1(p) X_M^{**}}{\mu_2(q) - \mu_1(p)} \tag{3.5}$$

and

$$\delta_n^* = \frac{X_M^{**} - X_N^*}{\mu_2(q) - \mu_1(p)} \tag{3.6}$$

The estimators (3.5) and (3.6) are simple linear functions of X_N^* and X_M^{**}, and hence functions of tails, depending on numbers p and q.

The asymptotic normality of X_N^* and X_M^{**} implies the asymptotic normality of the estimators λ_n^* and δ_n^*. Thus for large n we can use this fact to construct approximate confidence regions for the parameters λ and δ (cf. Cramér [6]).

Note that useful relationships between the tail means and variance will simplify subsequent calculations. Namely, it is easy to show that for a continuous random variable X with a finite variance Var $(X)=$ $=\sigma^2$ and the mean $E(X)=\mu$, for every p, $0<p<1$, we have

$$\mu = p\,\mu_1(p) + (1-p)\,\mu_2(p) \tag{3.7}$$

and

$$\sigma^2 = \mu\,\sigma_1^2(p) + (1-p)\,\sigma_2^2(p) + \\ + 2p\,(1-p)\,(\mu_2(p) - x_p)\,(x_p - \mu_1(p)) \tag{3.8}$$

where $x_p = F^{-1}(p)$. In addition, if the variable X is symmetrically distributed about the origin we have

$$\mu_1(p) = -\mu_2(1-p) \tag{3.9}$$

and

$$\sigma_2^2(1-p) = \sigma_1^2(p) \tag{3.10}$$

for every p, $0<p<1$.

(b) We extend the estimation procedure developed in the previous part by utilizing the mean of the middle portion of the ordered observations.

Let $0<p<q<1$. The numbers N and M then determine the partition of the ordered sample as follows:

$$\underbrace{X_{(1)} \leqslant \ldots \leqslant X_{(N)}}_{\text{left tail}} \leqslant \underbrace{X_{(N+1)} \leqslant \ldots \leqslant X_{(M)}}_{\text{middle part}} \leqslant \underbrace{X_{(M+1)} \leqslant \ldots \leqslant X_{(n)}}_{\text{right tail}}$$

Consider now linear estimators of λ and δ of the form

$$\tilde{\lambda}_n = c_1 X_N^* + c_2 \bar{X}_{N,M} + c_3 X_M^{**} \tag{3.11}$$

and

$$\tilde{\delta}_n = d_1 X_N^* + d_2 \bar{X}_{N,M} + d_3 X_M^{**} \tag{3.12}$$

where X_N^* and X_M^{**} are defined by (3.1) and (3.2) respectively and

$$\tilde{X}_{N,M} = \frac{1}{M-N} \sum_{j=N+1}^{M} X_{(j)} \tag{3.13}$$

The coefficients c_i, d_i, $i = 1, 2, 3$, are chosen in such a way that $\tilde{\lambda}_n$ and $\tilde{\delta}_n$ are asymptotically unbiased and have the smallest generalized variance among all unbiased estimators of the above form. The determination of these coefficients can be carried out very efficiently by the least squares method.

Theorems 4 and 5 give us the asymptotic expectations, variance and covariances of X_N^*, X_M^{**} and $\tilde{X}_{N,M}$. The expectations depend linearly, with known coefficients $\mu_1(p)$, $\mu_2(q)$, $\mu_3(p, q)$, on the unknown parameters λ and δ, and the variances and covariances are known up to a scalar factor δ^2.

All the conditions therefore exist for the application of the generalized least squares theorem (cf. Lloyd: "Generalized Least-Square Theorem" in Sarhan & Greenberg [13, pp. 20 - 27]).

Following Lloyd, written in matrix form, we have:

$$E(\underline{X}_0) = (\underline{1} : \underline{\alpha}) \begin{bmatrix} \lambda \\ \delta \end{bmatrix} ; \quad \mathrm{Var}(\underline{X}_0) \simeq \frac{\delta^2}{n} V \tag{3.14}$$

where for brevity we denote

$$\underline{X}_0 = \begin{bmatrix} X_N^* \\ \tilde{X}_{N,M} \\ X_M^{**} \end{bmatrix}, \quad \underline{1} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad \underline{\alpha} = \begin{bmatrix} \mu_1(p) \\ \mu_3(p, q) \\ \mu_2(q) \end{bmatrix}$$

and

$$V = \begin{bmatrix} \frac{\sigma_1^2(p)}{p} & \sigma_{13}(p, q) & \sigma_{12}(p, q) \\ \sigma_{13}(p, q) & \frac{\sigma_3^2(p, q)}{q - p} & \sigma_{23}(p, q) \\ \sigma_{12}(p, q) & \sigma_{23}(p, q) & \frac{\sigma_2^2(q)}{1 - q} \end{bmatrix} \tag{3.15}$$

Solving the normal equations

$$\left. \begin{aligned} (\underline{1}'\Omega\, \underline{1})\, \tilde{\lambda}_n + (\underline{1}'\Omega\, \underline{\alpha})\, \tilde{\delta}_n &= \underline{1}'\Omega\, \underline{X}_0 \\ (\underline{1}'\Omega\, \underline{\alpha})\, \tilde{\lambda}_n + (\underline{\alpha}'\, \Omega\, \underline{\alpha})\, \tilde{\delta}_n &= \underline{\alpha}'\, \Omega\, \underline{X}_0 \end{aligned} \right\} \tag{3.16}$$

we obtain the estimators

$$\tilde{\lambda}_n = -\underline{\alpha}'\, \Gamma\, \underline{X}_0 \tag{3.17}$$

$$\tilde{\delta}_n = \underline{1}'\, \Gamma\, \underline{X}_0 \tag{3.18}$$

where

$$\Omega = V^{-1}$$

$$\Gamma = \Omega\, (\underline{1}\, \underline{\alpha}' - \underline{\alpha}\, \underline{1}')\, \Omega / \Phi \tag{3.19}$$

and

$$\Phi = (\underline{1}'\, \Omega\, \underline{1})(\underline{\alpha}'\Omega\, \underline{\alpha}) - (\underline{1}'\, \Omega\, \underline{\alpha})^2 \tag{3.20}$$

Hence

$$\underline{c}' = [c_1\ c_2\ c_3] = -\underline{\alpha}'\, \Gamma \tag{3.21}$$

and

$$\underline{d}' = [d_1\ d_2\ d_3] = \underline{1}'\, \Gamma \tag{3.22}$$

Inasmuch as $\tilde{\lambda}_n$ and $\tilde{\delta}_n$ are least-square estimators of λ and δ, they are also the best linear estimators of λ and δ, based on X_N^*, $\tilde{X}_{N,M}$, and X_M^{**}.

Remark. The above least-squares method can be employed to obtain the estimators of λ and δ in the more general form:

$$\tilde{\lambda}_n = \sum_{i=1}^{r} c_i \tilde{X}_i, \qquad \tilde{\delta}_n = \sum_{i=1}^{r} d_i \tilde{X}_i$$

where $\tilde{X}_i$, $i = 1, ..., r$ can be any linear combinations of order statistics (including a single order statistic itself) which satisfy (3.14). So far, we have investigated procedures, based on X_N^*, $\tilde{X}_{N,M}$ and X_M^{**}, for jointly estimating unknown parameters λ and δ. If we assume that one of the parameters is known then the least-squares method can be adjusted by an obvious way to estimate the remaining unknown parameter. Such cases were considered in [10].

4. Efficiency

Following Cramér [6, p. 493], for a regular density $f_X \equiv \frac{1}{\delta} f\left(\frac{x-\lambda}{\delta}\right)$ we define the joint asymptotic relative efficiency (A.R.E.) of the estimators λ_n^* and δ_n^* as

$$e^*(p, q) = \lim_{n \to \infty} [n^2 \Delta G]^{-1} \tag{4.1}$$

where

$$G = \mathrm{Var}(\lambda_n^*)\,\mathrm{Var}(\delta_n^*) - \mathrm{Cov}^2(\lambda_n^*, \delta_n^*) \tag{4.2}$$

and

$$\Delta = E\left[\frac{\partial \log f_X}{\partial \lambda}\right]^2 E\left[\frac{\partial \log f_X}{\partial \delta}\right]^2 - E^2\left[\frac{\partial \log f_X}{\partial \lambda}\frac{\partial \log f_X}{\partial \delta}\right] \tag{4.3}$$

is the determinant of the information matrix corresponding to the density f_X. We note that in our case Δ can be further simplified yielding

$$\Delta = \frac{1}{\delta^4} \Delta_1 \tag{4.4}$$

where

$$\Delta_1 = E\left[\frac{f'(Z)}{f(Z)}\right]^2 E\left[1 + \frac{f'(Z)}{f(Z)} Z\right]^2 - E^2\left[\frac{f'(Z)}{f(Z)} \cdot \left[1 + \frac{f'(Z)}{f(Z)} Z\right]\right] \tag{4.5}$$

using for brevity $f' = df/dZ$ and $Z = \delta^{-1}(X - \lambda)$.

It can be shown easily that

$$G \simeq \frac{\delta^4}{n^2} \left\{ \frac{\dfrac{\sigma_1^2(p)\,\sigma_2^2(q)}{p\,(1-q)} - \sigma_{12}^2(p,q)}{[\mu_2(q) - \mu_1(p)]^2} \right\} \tag{4.6}$$

and hence

$$e^*(p,q) = \frac{1}{\Delta_1} \cdot \frac{[\mu_2(q) - \mu_1(p)]^2}{\left[\dfrac{\sigma_1^2(p)\,\sigma_2^2(q)}{p\,(1-q)} - \sigma_{12}^2(p,q)\right]} \tag{4.7}$$

Similarly we derive the formula for the joint A.R.E. of $\tilde{\lambda}_n$ and $\tilde{\delta}_n$, denoted here by $\tilde{e}(p, q)$. Using the notation of the previous section, for large n we have:

$$\text{Var}\,(\tilde{\lambda}_n) \simeq \frac{\delta^2}{n} \frac{\underline{\alpha}'\,\Omega\,\underline{\alpha}}{\Phi} \tag{4.8}$$

$$\text{Var}\,(\tilde{\delta}_n) \simeq \frac{\delta^2}{n} \frac{\underline{1}'\,\Omega\,\underline{1}}{\Phi} \tag{4.9}$$

and

$$\text{Cov}\,(\tilde{\lambda}_n, \tilde{\delta}_n) \simeq -\frac{\delta^2}{n} \frac{(\underline{1}'\,\Omega\,\underline{\alpha})}{\Phi} \tag{4.10}$$

Therefore, the joint A.R.E. of $\tilde{\lambda}_n$ and $\tilde{\delta}_n$ is

$$\tilde{e}(p, q) = \frac{\Phi}{\Delta_1} \tag{4.11}$$

The efficiencies $e^*(p, q)$ and $\tilde{e}(p, q)$ depend strongly on the numbers p and q, so one can look for the optimal values of p and q, say p^* and q^*, which maximize the above efficiencies.

5. Examples

In this section we illustrate the estimation procedures developed in Section 3 on a number of examples. Throughout this section all the required assumptions are satisfied and only complete samples are investigated. Right and left censored samples, treated in a similar manner, are considered elsewhere [10]. The search for optimum p^* and q^* was carried out very simply –tabulating efficiency for various values of p and q and then selecting the best possible combination.

Example 1 – Gumbel distribution

The estimators of λ and δ for the Gumbel distribution

$$F\left[\frac{x-\lambda}{\delta}\right] = \exp\left[-\exp\left[-\frac{x-\lambda}{\delta}\right]\right], \quad -\infty < x < \infty \tag{5.1}$$

are determined by (3.5) and (3.6) respectively, where we have

$$\mu_1(p) = \frac{1}{p}\int_{-\infty}^{x_p} x\, e^{-x}\, e^{-e^{-x}}\, dx \tag{5.2}$$

and

$$\mu_2(q) = \frac{1}{1-q}\int_{x_q}^{\infty} x\, e^{-x}\, e^{-e^{-x}}\, dx \tag{5.3}$$

with the notation $x_p = -\log(-\log(p))$ and $x_q = -\log(-\log(q))$. The integrals require numerical integration and the values of $\mu_1(p)$ and $\mu_2(q)$ are given in Table 1. The values of $\sigma_1^2(p)$, $\sigma_2^2(q)$ and $\sigma_{12}(p, q)$ are given in [10]. The determinant of the information matrix, Δ, was obtained by Tiago de Oliveira [14] and is $\Delta = \pi^2/6\ \delta^4$. The joint A.R.E. of λ_n^* and δ_n^*, $e^*(p, q)$, was calculated for values of $p = 0.05\ (0.05)\ 0.95$ and $q = 0.05\ (0.05)\ 0.95$, and is given in Table 2.

Clearly, the efficiency attains its maximum at the point $p^* = .2$ and $q^* = .1$, where $e^*(p^*, q^*) = 89\%$. In this case the sample tails overlap and the "most efficient" estimates of λ and δ based on sample tails are

$$\left.\begin{aligned} \lambda_n^* &= 0.461\ X_N^* + 0.539\ X_M^{**} \\ \delta_n^* &= 0.598\ (X_M^{**} - X_N^*) \end{aligned}\right\} \qquad (5.4)$$

where X_N^* is the left sample tail mean based on the smallest 20% of the ordered observations, and X_M^{**} the right sample tail mean based on the largest 90% of the ordered observations. We get practically no difference in efficiency considering the non-overlapping case $p = q = 0.2$; here $e^*(0.2,\ 0.2) = 88\%$. In this case we use X_N^*, the left sample tail mean, based on the smallest 20% of the ordered observations and X_M^{**}, the right sample tail mean based on the largest 80% of the ordered observations. In other words, we divide the ordered sample into two contiguous non-overlapping groups.

Consider now the estimators $\tilde{\lambda}_n$ and $\tilde{\delta}_n$. The truncated mean $\mu_3(p, q)$ can be obtained using the representation

$$\mu_3(p, q) = \frac{1}{q - p}\,[q\,\mu_1(q) - p\,\mu_1(p)] \qquad (5.5)$$

The values of the joint A.R.E., $\tilde{e}(p, q)$, of the estimators $\tilde{\lambda}_n$ and $\tilde{\delta}_n$ for the following values of p and q: $p = 0.05\ (0.05)\ 0.90$, $q = p + 0.05\ (0.05)\ 0.95$ are given in Table 3. From Table 3, it can be seen that the efficiency attains its maximum at the point $p^* = 0.05$ and $q^* = 0.40$, where $\tilde{e}(0.05,\ 0.40) = 97\%$. Recall that the best estimate based only on sample tails has an efficiency of about 89%.

For $p^* = 0.05$ and $q^* = 0.40$ we find

$$c_1 = 0.12 \qquad d_1 = -0.27$$
$$c_2 = 0.57 \qquad d_2 = -0.15$$
$$c_3 = 0.31 \qquad d_3 = \ \ 0.42$$

Hence the best estimators of λ and δ based on means of three contiguous groups are

$$\tilde{\lambda}_n = 0.12\, X_N^* + 0.57\, \tilde{X}_{N,M} + 0.31\, X_M^{**}$$

and

$$\tilde{\delta}_n = -0.27\, X_N^* - 0.15\, \tilde{X}_{N,M} + 0.42\, X_M^{**}$$

where X_N^* is the sample mean of the smallest 5% of the order statistics, X_M^{**} is the sample mean of the largest 60% of the order statistics, and $X_{N,M}^{**}$ is the sample mean of the remaining 35% of the order statistics.

Example 2 – Normal distribution

To conform with standard notation, we let $\lambda \equiv \mu$, and $\delta \equiv \sigma$. In this example the estimators based on sample tails of μ and σ for the normal distribution $N(\mu, \sigma^2)$ are:

$$\mu_n^* = \frac{\mu_2(q)\, X_N^* - \mu_1(p)\, X_M^{**}}{\mu_2(q) - \mu_1(p)} \tag{5.6}$$

and

$$\sigma_n^* = \frac{X_M^{**} - X_N^*}{\mu_2(q) - \mu_1(p)} \tag{5.7}$$

The means $\mu_1(p)$ and $\mu_2(q)$ can be easily calculated, since

$$\mu_1(p) = \frac{1}{p} \int_{-\infty}^{x_p} x \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\, dx = -\frac{1}{p\sqrt{2\pi}} e^{-x_p^2/2} \tag{5.8}$$

where x_p denotes the p- quantile of $N(0, 1)$. By symmetry considerations we have

$$\mu_2(q) = -\mu_1(1-q)$$

The computation of $\sigma_1^2(p)$, $\sigma_2^2(q)$ and $\sigma_{12}(p, q)$ is also not difficult.

The values of the joint A.R.E., $e^*(p, q)$, of the estimators μ^* and σ^*, given by (4.7), where $\Delta_1 = 2$, are reproduced in Table 4 for various values of p and q: $p = 0.05\ (0.05)\ 0.95$ and $q = 0.05\ (0.05)\ 0.95$.

From Table 4, it can be seen that the maximum efficiency is attained at $P^* = 0.8$; $q^* = 0.2$, where $e^*(p^*, q^*) = 95\%$ (overlapping case). At this point the estimators (5.6) and (5.7) become

$$\mu_n^* = \frac{1}{2}(X_N^* + X_M^{**})$$

and

$$\sigma_n^* = 1.43\ (X_M^{**} - X_N^*)$$

where X_N^*, the left sample tail mean, is based on the smallest 80% of the ordered observations and X_M^{**}, the right sample tail mean, on the largest 80% of the ordered observations.

It is interesting to note that in the neighbourhood of the point $p = q = 0.5$, where we have $e^*(0.5, 0.5) = 88\%$, the gradients of the efficiency in p and q are very small. Even for $p = 0.4$ and $q = 0.6$ we have $e^*(0.4, 0.6) = 88\%$. In this case we do not use 20% of the middle part of the ordered data, and the estimators coincide with those of Abe [2].

The joint A.R.E. of the estimators $\tilde{\mu}_n$ and $\tilde{\sigma}_n$, based on three contiguous blocks of ordered observations, is given in Table 5. The efficiency, only slightly improved, attains its maximum at the point $p^* = 0.2$ and $q^* = 0.8$, where we have $\tilde{e}(0.2, 0.8) = 97\%$.

For $p^* = 0.2$ and $q^* = 0.8$, we obtain

$$c_1 = 0.2 \qquad d_1 = -0.36$$

$$c_2 = 0.6 \qquad d_2 = \ \ 0.0$$

$$c_3 = 0.2 \qquad d_3 = \ \ 0.36$$

and then
$$\tilde{\mu}_n = 0.2\, X_N^* + 0.6\, \tilde{X}_{N,M} + 0.2\, X_M^{**} = \bar{X}_n$$

$$\tilde{\sigma}_n = 0.36\, (X_M^{**} - X_N^*)$$

where X_N^* is the sample mean of the smallest 20% and X_M^{**} is the sample mean of the largest 20% of the order statistics.

Clearly, $\tilde{\mu}_n$ is the sample mean and an efficient estimator of μ. The location-invariant estimator $\tilde{\sigma}_n$ coincides with the estimator proposed by D'Agostino & Cureton [7], and by Abe [2], whose efficiency was found also to be 97%.

Example 3 – Logistic distribution

We consider the estimators of λ and δ determined by (3.5) and (3.6) for the logistic distribution having c.d.f.

$$F\left[\frac{x-\lambda}{\delta}\right] = \left[1 + \exp\left\{-\frac{x-\lambda}{\delta}\right\}\right]^{-1}, \quad -\infty < x < \infty \qquad (5.9)$$

The left and right tail means $\mu_1(p)$ and $\mu_2(q)$ are respectively:

$$\mu_1(p) = \frac{1}{p}\log(1-p) - \log\left[\frac{1-p}{p}\right] \qquad (5.10)$$

and

$$\mu_2(q) = -\mu_1(1-q) = -\frac{1}{1-q}\log q + \log\left[\frac{q}{1-q}\right] \qquad (5.11)$$

The computation of $\sigma_1^2(p)$ and $\sigma_{12}(p, q)$, for $p > q$, requires a numerical integration.

The values of the joint A.R.E. of the estimators λ_n^* and δ_n^* are given in Table 6. From Table 6 it can be seen that the maximum efficiency is attained at $p^* = 0.7$, $q^* = 0.3$, where we have: $e^*(p^*, q^*) = 96\%$. At this point the estimators λ_n^* and δ_n^* become

$$\lambda_n^* = \frac{1}{2}(X_N^* + X_M^{**})$$

$$\delta_n^* = 0.56\,(X_M^{**} - X_N^*)$$

where X_N^* is the average of the smallest 70% of the ordered observations and X_M^{**} is the average of the largest 70% of the ordered observations. Note that in the neighbourhood of p^* and q^* the efficiency is again very insensitive to changes in p and q.

Considering only non-overlapping tails, we obtain the best efficiency at $p = q = 0.5$, where $e^*(0.5, 0.5) = 90\%$. In this case X_N^* and X_M^{**} are the averages of the smallest 50% and largest 50% of the ordered data.

In Table 7, the values of the joint A.R.E., $\bar{e}(p, q)$, of the estimators $\bar{\lambda}_n$ and $\bar{\delta}_n$ are shown. Clearly, the efficiency attains its maximum around the point $p^* = 0.3$, $q^* = 0.7$ where we have $\bar{e}(p, q) = 98\%$. For $p^* = 0.3$, $q^* = 0.7$ we get

$$\begin{aligned} c_1 &= .15 & d_1 &= -0.25 \\ c_2 &= .70 & d_2 &= 0.0 \\ c_3 &= .15 & d_3 &= 0.25 \end{aligned}$$

and hence

$$\bar{\lambda}_n = 0.15\,X_N^* + 0.7\,\bar{X}_{N,M} + 0.15\,X_M^{**}$$

$$\bar{\delta}_n = 0.25\,(X_M^{**} - X_N^*)$$

where X_N^* and X_M^{**} are the sample means of the smallest and largest 30% of the ordered data, respectively. $\bar{X}_{N,M}$ is the sample mean of the remaining 40% of the ordered observations.

Example 4 — Weibull distribution

In the last example we shall consider the Weibull distribution with the shape parameter β and the scale parameter θ, corresponding to a random variable W, having the c.d.f.

$$F_W(x;\beta,\theta) = 1 - \exp\left\{-\left[\frac{x}{\theta}\right]^{\beta}\right\}; \quad x \geqslant 0 \qquad (5.12)$$

$$= 0 \qquad ; \quad x < 0$$

for $\beta > 0$, and $\theta > 0$. This distribution is not a member of a scale-location distribution family, but it is well known that the random variable $X = -\log W$ has the Gumbel distribution (5.1) with parameters $\lambda = -\log\theta$ and $\delta = \beta^{-1}$.

Let $W_{(1)} \leqslant \ldots \leqslant W_{(n)}$ denote an ordered sample of size n from the Weibull distribution with c.d.f. (5.12).

We define the statistics W_N^* and W_M^{**} as follows:

$$W_N^* = \frac{1}{N}\sum_{j=1}^{N} -\log(W_{(j)})$$

and

$$W_M^{**} = \frac{1}{n-M}\sum_{j=M+1}^{n} -\log(W_{(j)})$$

or equivalently

$$W_N^* = -\log\left[\prod_{i=1}^{N} W_{(j)}\right]^{1/N} \qquad (5.13)$$

and

$$W_M^{**} = -\log\left[\prod_{i=M+1}^{n} W_{(j)}\right]^{1/n-M} \qquad (5.14)$$

By Theorem 2, W_N^* and W_M^{**} will converge in probability to

$$\mu_1^*(p) = \frac{1}{p} \int_0^p g(s)\, ds \tag{5.15}$$

and

$$\mu_2^*(q) = \frac{1}{1-q} \int_q^1 g(s)\, ds \tag{5.16}$$

respectively, as $n \to \infty$, where $g(s)$ is now defined as

$$g(s) = -\log (F_W^{-1}(s)) = -\log \theta - \frac{1}{\beta} \log (-\log (1-s)) \tag{5.17}$$

and $F_W^{-1}(s)$ denotes the s- quantile of (5.12).

We obtain

$$\mu_1^*(p) = \frac{1}{\beta} \mu_2 (1-p) - \log \theta \tag{5.18}$$

and

$$\mu_2^*(q) = \frac{1}{\beta} \mu_1 (1-q) - \log \theta \tag{5.19}$$

where $\mu_1(1-q)$ and $\mu_2(1-p)$ are already given by equations (5.2) and (5.3) respectively, and their values can be read from Table 1.

Then, it is easy to show that our asymptotically unbiased estimators, based on tails, of the parameters θ and β, are:

$$\theta_n^* = \exp \left\{ - \frac{W_M^{**} \mu_2 (1-p) - W_N^* \mu_1 (1-q)}{\mu_2 (1-p) - \mu_1 (1-p)} \right\} \tag{5.20}$$

$$\beta_n^* = \frac{\mu_2 (1-p) - \mu_1 (1-q)}{W_N^* - W_M^{**}} \tag{5.21}$$

Let us denote by $e_W^*(p, q)$ the joint A.R.E. of θ_n^* and β_n^*. Not surprisingly, since the transformation $-\log(\cdot)$ reverses the order of observations, we get:

$$e_W^*(p, q) = e^*(1-p, 1-q) \tag{5.22}$$

where $e^*(\cdot,\cdot)$ is the A.R.E. of λ_n^* and δ_n^* in the Gumbel case. In fact we have: $W_N^* = X_{n-N}^{**}$ and $W_M^{**} = X_{n-M}^*$. Thus the maximum efficiency is also 89%, reached at the points $p^* = 0.9$, $q^* = 0.8$.

The "reversed" results also hold for the case of three contiguous groups of ordered observations.

6. Concluding remarks

(a) A question arising in practical applications concerns the determination of a reasonable sample size n, as required for successfull application of an asymptotic method.

For the Gumbel distribution this question has been investigated in more detail in [10]. Random samples of X_N^* and X_N^{**} (for $p = q = .2$) were generated for various values of n and it was found that for sample sizes close to sixty the distributions of X_N^* and X_M^{**}, and hence of λ_n^* and δ_n^*, can be very well approximated by the normal distribution.

The estimators λ_n^* and $\tilde{\lambda}_n$ are unbiased for any sample size n, provided that the underlying distribution is symmetric and the same number of observations is used in each tail. In this case, since $E(X_N^*) \uparrow \mu_1(p)$ (and $E(X_{n-N}^{**}) \downarrow \mu_2(1-p)$), the estimators δ_n^* and $\tilde{\delta}_n$ underestimate the scale parameter δ.

For instance, if $n = 20$ and the underlying distribution is normal, we have $N = 16$, $M = 4$ (optimum spacing for σ_n^* is $p^* = 1 - q^* = 0.8$) and the unbiased estimator of σ is then

$$\sigma_{20}^* = 1.5018\,(X_4^{**} - X_{16}^*) \tag{6.1}$$

since $E(X_{16}^*) = -0.3329$. The expectation $E(X_{16}^*)$ is computed using the table of expected values of order statistics (cf. Teichroew in Sarhan & Greenberg [13, p. 193]). For comparison, in Example 2 we derived $\mu_1(.8) = -0.3499$ and $\sigma_n^* = 1.4290\ (X_M^{**} - X_N^*)$. It seems to us that by introducing a correction factor $n/(n-1)$ even this small bias could be practically eliminated. For small sample sizes we suggest replacing σ_n^*, given by (5.7), by

$$\bar{\sigma}_n^* = \left[\frac{n}{n-1}\right] \frac{X_M^{**} - X_N^*}{\mu_2(q) - \mu_1(p)} \tag{6.2}$$

Thus, for $n = 20$ and $p^* = 1 - q^* = 0.8$ we get $\bar{\sigma}_{20}^* = 1.5041(X_4^{**} - X_{16}^*)$. Clearly the difference between σ_{20}^* and $\bar{\sigma}_{20}^*$ is very small.

(b) If an underlying distribution is symmetric and $q = 1 - p$, i.e. symmetric tails are used, then it can be shown that λ_n^* and δ_n^* are asymptotically uncorrelated and therefore asymptotically independent. In this case, the construction of confidence intervals is simple. We also note that for symmetric distributions the optimal selection of tails is also symmetric, i.e. $q^* = 1 - p^*$. The same conclusions are also true for $\tilde{\lambda}_n$ and $\tilde{\delta}_n$.

(c) The estimators examined in the preceding sections are special cases of a large class of estimators having a form

$$\tilde{v}_n = \sum_{i=1}^{n} c_i X_{(i)} \tag{6.3}$$

It has been shown that for $c_i = J\left(\frac{i}{n+1}\right)$, where $J(\cdot)$ is a suitably selected weight function, $\tilde{v}_n$ became asymptotically efficient [5]. Therefore, our estimators utilizing the means of optimally chosen blocks of order statistics may be regarded as approximations to (6.3). In fact, we approximate the optimal weight function $J(\cdot)$ by an optimally selected step function.

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Table 1 Percentage points and tail means for the Gumbel distribution.

p	x_p	$\mu_1(p)$	$\mu_2(p)$
0.02	-1.364	-1.574	0.621
0.04	-1.169	-1.416	0.660
0.06	-1.034	-1.310	0.698
0.08	-0.927	-1.227	0.734
0.10	-0.834	-1.158	0.770
0.12	-0.752	-1.097	0.805
0.14	-0.676	-1.042	0.841
0.16	-0.606	-0.992	0.876
0.18	-0.539	-0.945	0.911
0.20	-0.476	-0.901	0.947
0.22	-0.415	-0.860	0.983
0.24	-0.356	-0.820	1.018
0.26	-0.298	-0.782	1.055
0.28	-0.241	-0.746	1.092
0.30	-0.186	-0.710	1.129
0.32	-0.131	-0.676	1.167
0.34	-0.076	-0.642	1.205
0.36	-0.021	-0.609	1.245
0.38	0.033	-0.577	1.284
0.40	0.087	-0.545	1.325
0.42	0.142	-0.513	1.367
0.44	0.197	-0.482	1.410
0.46	0.253	-0.452	1.454
0.48	0.309	-0.421	1.499
0.50	0.367	-0.391	1.545
0.52	0.425	-0.361	1.593
0.54	0.484	-0.330	1.643
0.56	0.545	-0.300	1.694
0.58	0.607	-0.270	1.747
0.60	0.672	-0.240	1.803
0.62	0.738	-0.209	1.860
0.64	0.807	-0.179	1.921
0.66	0.878	-0.148	1.984
0.68	0.953	-0.116	2.051
0.70	1.031	-0.085	2.122
0.72	1.113	-0.053	2.197
0.74	1.200	-0.020	2.277
0.76	1.293	0.013	2.363
0.78	1.392	0.047	2.455
0.80	1.500	0.082	2.556
0.82	1.617	0.118	2.667
0.84	1.747	0.156	2.791
0.86	1.892	0.194	2.930
0.88	2.057	0.235	3.089
0.90	2.250	0.277	3.277
0.92	2.484	0.323	3.505
0.94	2.783	0.372	3.798
0.96	3.199	0.426	4.209
0.98	3.902	0.489	4.908

Table 2 Joint A.R.E. $e^*(p,q)$ of the estimators λ_n^* and δ_n^*--Gumbel distribution

	Q = 0.05	0.10	0.15	0.20	0.25	0.30	0.35	0.40	0.45	0.50	0.55	0.60	0.65	0.70	0.75	0.80	0.85	0.90	0.95
P = 0.05	67	68	68	68	68	67	66	65	64	62	60	58	55	52	48	43	37	30	19
P = 0.10	31	81	81	81	81	80	80	79	77	76	74	71	68	64	59	54	47	38	25
P = 0.15	87	86	86	86	86	86	85	84	83	81	79	76	73	69	65	59	51	42	28
P = 0.20	89	89	88	88	87	87	86	86	84	83	81	78	75	71	67	61	53	43	29
P = 0.25	89	89	88	88	87	87	86	85	84	83	81	78	75	72	67	61	54	44	30
P = 0.30	88	88	88	87	86	85	85	84	83	81	79	77	74	71	66	61	54	44	30
P = 0.35	86	86	86	86	85	84	83	82	81	79	77	75	73	69	65	60	53	43	30
P = 0.40	83	84	84	84	83	82	81	79	78	77	75	73	70	67	63	58	51	42	29
P = 0.45	80	81	82	81	81	80	78	77	75	74	72	70	68	65	61	56	50	41	29.
P = 0.50	76	78	79	79	78	77	76	75	73	71	69	67	65	62	58	54	48	40	28
P = 0.55	72	75	76	76	76	75	74	72	71	69	67	64	62	59	56	51	46	38	27
P = 0.60	68	71	73	74	73	73	71	70	68	66	64	62	59	56	53	49	44	36	25
P = 0.65	64	68	70	71	71	70	69	68	66	64	62	59	56	54	50	46	41	34	24
P = 0.70	59	64	66	68	68	68	67	66	64	62	59	57	54	51	48	44	39	32	23
P = 0.75	54	59	63	65	66	66	65	64	62	60	57	55	52	49	45	41	37	30	21
P = 0.80	49	55	59	62	63	64	63	62	60	58	56	53	50	47	43	39	34	28	20
P = 0.85	43	51	56	59	61	62	62	61	59	57	55	52	49	45	41	37	32	26	19
P = 0.80	37	46	53	57	60	61	61	61	59	57	54	51	48	44	40	36	30	25	17
P = 0.95	31	43	52	58	62	63	64	63	61	59	56	53	49	45	40	35	30	23	16

Table 3 Joint asymptotic relative efficiency $\tilde{e}(p,q)$, of the estimators $\tilde{\lambda}_n$ and $\tilde{\delta}_n$--Gumbel distribution

	Q = 0.10	0.15	0.20	0.25	0.30	0.35	0.40	0.45	0.50	0.55	0.60	0.65	0.70	0.75	0.80	0.85	0.90	0.95
P = 0.05	81	88	92	94	96	97	97	97	96	95	94	92	90	88	86	83	80	76
P = 0.10	**	87	90	92	94	95	96	96	96	96	96	95	94	93	92	90	89	86
P = 0.15	**	**	89	90	92	93	93	94	94	95	95	94	94	94	93	92	91	90
P = 0.20	**	**	**	89	90	90	91	91	92	92	92	92	92	92	92	92	91	90
P = 0.25	**	**	**	**	88	88	88	89	89	89	90	90	90	90	90	90	89	89
P = 0.30	**	**	**	**	**	86	86	86	87	87	87	87	87	87	87	87	87	87
P = 0.35	**	**	**	**	**	**	84	84	84	84	84	84	84	84	84	84	84	84
P = 0.40	**	**	**	**	**	**	**	82	81	81	81	81	80	80	80	80	80	80
P = 0.45	**	**	**	**	**	**	**	**	79	78	77	77	77	77	77	76	76	76
P = 0.50	**	**	**	**	**	**	**	**	**	75	74	74	73	73	73	73	72	72
P = 0.55	**	**	**	**	**	**	**	**	**	**	71	71	70	69	69	68	68	68
P = 0.60	**	**	**	**	**	**	**	**	**	**	**	67	66	65	65	64	63	63
P = 0.65	**	**	**	**	**	**	**	**	**	**	**	**	63	61	60	60	59	58
P = 0.70	**	**	**	**	**	**	**	**	**	**	**	**	**	58	56	55	54	53
P = 0.75	**	**	**	**	**	**	**	**	**	**	**	**	**	**	52	50	49	47
P = 0.80	**	**	**	**	**	**	**	**	**	**	**	**	**	**	**	45	43	42
P = 0.85	**	**	**	**	**	**	**	**	**	**	**	**	**	**	**	**	38	36
P = 0.90	**	**	**	**	**	**	**	**	**	**	**	**	**	**	**	**	**	29

Table 4 Joint A.R.E. $e^*(p,q)$ of the estimators μ_n^* and σ_n^*--Normal distribution

	Q = 0.05	0.10	0.15	0.20	0.25	0.30	0.35	0.40	0.45	0.50	0.55	0.60	0.65	0.70	0.75	0.80	0.85	0.90	0.95
P = 0.05	42	43	45	46	47	48	49	49	50	50	50	49	48	47	45	42	38	32	23
P = 0.10	56	57	58	60	61	63	64	65	66	66	66	66	65	64	61	58	53	45	32
P = 0.15	65	65	66	68	69	71	72	74	75	75	76	75	75	73	71	67	61	53	38
P = 0.20	72	72	72	73	74	76	77	79	80	81	81	81	80	79	77	73	67	58	42
P = 0.25	78	77	77	77	78	79	81	82	83	84	85	85	84	83	80	77	71	61	45
P = 0.30	82	81	81	81	81	82	83	84	85	86	87	87	86	85	83	79	73	64	47
P = 0.35	85	84	84	83	83	84	84	85	86	87	88	88	87	86	84	80	75	65	48
P = 0.40	88	87	86	86	85	85	86	86	87	87	88	88	88	87	85	81	75	66	49
P = 0.45	90	89	88	88	87	87	87	87	87	88	88	88	88	87	85	81	76	66	50
P = 0.50	91	91	90	90	89	88	88	88	88	88	88	87	87	86	84	81	75	66	60
P = 0.55	92	92	92	91	90	90	89	89	88	88	87	87	86	85	83	80	75	66	50
P = 0.60	92	93	93	92	92	91	90	89	89	88	87	86	85	84	82	79	74	65	49
P = 0.65	92	94	94	93	93	92	91	90	89	88	87	86	84	83	81	77	72	64	49
P = 0.70	92	94	94	94	93	93	92	91	90	88	87	85	84	82	79	76	71	63	48
P = 0.75	90	93	94	95	94	93	93	92	90	89	87	85	83	81	78	74	69	61	47
P = 0.80	88	92	94	95	95	94	93	92	91	90	88	86	83	81	77	73	68	60	46
P = 0.85	85	91	93	94	94	94	94	93	92	90	88	86	84	81	77	72	66	58	45
P = 0.90	80	87	91	92	93	94	94	93	92	91	89	87	84	81	77	72	65	57	43
P = 0.95	72	80	85	88	90	92	92	92	92	91	90	88	85	82	73	72	65	56	42

Table 5 Joint asymptotic relative efficiency $\tilde{e}(p,q)$ of the estimators $\tilde{\mu}_n$ and $\tilde{\sigma}_n$--Normal distribution

	Q = 0.10	0.15	0.20	0.25	0.30	0.35	0.40	0.45	0.50	0.55	0.60	0.65	0.70	0.75	0.80	0.85	0.90	0.95
P = 0.05	59	67	73	78	82	86	88	91	92	93	94	94	94	92	90	87	81	72
P = 0.10	**	68	74	78	82	85	88	90	92	94	95	95	96	95	94	92	88	81
P = 0.15	**	**	75	78	82	85	87	90	92	93	95	96	96	96	96	95	92	87
P = 0.20	**	**	**	79	82	84	87	89	91	93	94	95	96	97	97	96	94	90
P = 0.25	**	**	**	**	82	84	86	88	90	92	93	94	95	96	97	96	95	92
P = 0.30	**	**	**	**	**	85	86	88	89	91	92	94	95	95	96	96	96	94
P = 0.35	**	**	**	**	**	**	86	87	89	90	91	92	94	94	95	96	95	94
P = 0.40	**	**	**	**	**	**	**	87	88	89	90	91	92	93	94	95	95	94
P = 0.45	**	**	**	**	**	**	**	**	88	88	89	90	91	92	93	93	94	93
P = 0.50	**	**	**	**	**	**	**	**	**	88	88	89	90	91	92	92	92	92
P = 0.55	**	**	**	**	**	**	**	**	**	**	87	87	88	88	89	90	90	91
P = 0.60	**	**	**	**	**	**	**	**	**	**	**	86	86	86	87	87	88	88
P = 0.65	**	**	**	**	**	**	**	**	**	**	**	**	85	84	84	85	85	86
P = 0.70	**	**	**	**	**	**	**	**	**	**	**	**	**	82	82	82	82	82
P = 0.75	**	**	**	**	**	**	**	**	**	**	**	**	**	**	79	78	78	78
P = 0.80	**	**	**	**	**	**	**	**	**	**	**	**	**	**	**	75	74	73
P = 0.85	**	**	**	**	**	**	**	**	**	**	**	**	**	**	**	**	68	67
P = 0.90	**	**	**	**	**	**	**	**	**	**	**	**	**	**	**	**	**	59

Table 6 Joint A.R.E. $e^*(p,q)$ of the estimators λ_n^* and δ_n^*--Logistic distribution

	Q = 0.05	0.10	0.15	0.20	0.25	0.30	0.35	0.40	0.45	0.50	0.55	0.60	0.65	0.70	0.75	0.80	0.85	0.90	0.95
P = 0.05	31	32	33	34	34	34	34	34	33	32	31	30	28	26	23	21	17	13	8
P = 0.10	46	47	48	49	50	50	50	50	49	48	47	45	43	40	36	32	27	21	13
P = 0.15	57	57	58	59	60	61	61	61	60	59	58	56	53	50	46	41	35	27	17
P = 0.20	66	66	67	68	68	69	69	69	69	68	66	64	61	58	53	48	41	32	21
P = 0.25	73	73	73	74	75	75	76	76	75	74	73	71	68	64	59	53	46	36	23
P = 0.30	79	79	79	79	80	80	80	80	80	79	78	76	73	69	64	58	50	40	26
P = 0.35	83	83	84	84	84	84	84	84	84	83	81	79	76	73	68	61	53	43	28
P = 0.40	87	87	87	88	88	88	88	87	87	86	84	82	79	76	71	64	56	45	30
P = 0.45	89	90	90	90	91	91	90	90	89	88	86	84	81	78	73	66	58	47	31
P = 0.50	91	92	92	93	93	93	92	92	91	90	88	86	83	79	74	68	59	48	32
P = 0.55	92	93	94	94	94	94	94	93	92	91	89	87	84	80	75	69	60	49	33
P = 0.60	91	93	94	95	96	96	95	94	93	92	90	87	84	80	76	69	61	50	34
P = 0.65	90	93	94	95	96	96	96	95	94	92	90	88	84	80	76	69	61	50	34
P = 0.70	88	91	93	95	96	96	96	96	94	93	91	88	84	80	75	69	61	50	34
P = 0.75	84	89	92	94	95	96	96	96	94	93	91	88	84	80	75	68	60	50	34
P = 0.80	79	85	89	92	94	95	95	95	94	93	90	88	84	79	74	68	59	49	34
P = 0.85	73	80	85	89	92	93	94	94	94	92	90	87	84	79	73	67	58	48	33
P = 0.90	65	74	80	85	89	91	93	93	93	92	90	87	83	79	73	66	57	47	32
P = 0.95	53	65	73	79	84	88	90	91	92	91	89	87	83	79	73	66	57	46	31

Table 7 Joint asymptotic relative efficiency $\tilde{e}(p,q)$, of the estimators $\tilde{\lambda}_n$ and $\tilde{\delta}_n$--Logistic distribution

	Q = 0.10	0.15	0.20	0.25	0.30	0.35	0.40	0.45	0.50	0.55	0.60	0.65	0.70	0.75	0.80	0.85	0.90	0.95
P = 0.05	54	64	72	78	83	87	90	92	94	94	93	92	89	86	81	75	67	55
P = 0.10	**	67	74	80	85	88	91	93	95	96	95	95	93	91	87	82	76	67
P = 0.15	**	**	76	81	85	89	92	94	95	96	97	96	95	94	91	87	82	75
P = 0.20	**	**	**	82	86	89	92	94	96	97	97	97	97	96	94	91	87	81
P = 0.25	**	**	**	**	87	90	92	94	96	97	97	98	98	97	96	94	91	86
P = 0.30	**	**	**	**	**	90	92	94	95	97	97	98	98	98	97	95	93	89
P = 0.35	**	**	**	**	**	**	92	94	95	96	97	98	98	98	97	96	95	92
P = 0.40	**	**	**	**	**	**	**	94	95	96	96	97	97	97	97	97	95	93
P = 0.45	**	**	**	**	**	**	**	**	94	95	96	96	97	97	97	96	96	94
P = 0.50	**	**	**	**	**	**	**	**	**	94	95	95	95	96	96	95	95	94
P = 0.55	**	**	**	**	**	**	**	**	**	**	94	94	94	94	94	94	93	92
P = 0.60	**	**	**	**	**	**	**	**	**	**	**	92	92	92	92	92	91	90
P = 0.65	**	**	**	**	**	**	**	**	**	**	**	**	90	90	89	89	88	87
P = 0.70	**	**	**	**	**	**	**	**	**	**	**	**	**	87	86	85	85	83
P = 0.75	**	**	**	**	**	**	**	**	**	**	**	**	**	**	82	81	80	78
P = 0.80	**	**	**	**	**	**	**	**	**	**	**	**	**	**	**	76	74	72
P = 0.85	**	**	**	**	**	**	**	**	**	**	**	**	**	**	**	**	67	64
P = 0.90	**	**	**	**	**	**	**	**	**	**	**	**	**	**	**	**	**	54